PHYSICAL REVIEW B **78**, 054436 (2008)

Spin-orbital effects in magnetized quantum wires and spin chains

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(Received 5 June 2008; published 25 August 2008)

We present analysis of the interacting quantum wire problem in the presence of magnetic field and spin-orbital interaction. We show that an interesting interplay of Zeeman and spin-orbit terms, facilitated by the electron-electron interaction, results in the spin-density wave (SDW) state when the magnetic-field and spin-orbit axes are *orthogonal*. We show that this instability is enhanced in a closely related problem of Heisenberg spin chain with asymmetric uniform Dzyaloshinskii-Moriya (DM) interaction. Magnetic field in the direction perpendicular to the DM anisotropy axis results in staggered long-range magnetic order along the orthogonal to the applied field direction. We explore consequences of the uniform DM interaction for the electron-spin-resonance (ESR) measurements, and point out that they provide way to probe right- and left-moving excitations of the spin chain separately.

DOI: 10.1103/PhysRevB.78.054436 PACS number(s): 71.70.Ej, 73.63.Nm, 75.10.Pq

I. INTRODUCTION

Over the last several years there has been a remarkable growth in research activity in the field of spintronics with the ultimate goal to fabricate novel spin-filter devices which can control and manipulate the electron spins.[1] The proposals for such a spin-filter device contains two achievable attributes, a ballistic quantum wire and the presence of a tunable Rashba spin-orbit coupling responsible for controlling the electron spin. Ballistic quantum wires are created in a two-dimensional electron gas (2DEG) by cleaved edge over growth, whereas the Rashba effect arises due to the asymmetry associated with the confinement potential.[2] The asymmetry and hence the Rashba coupling strength can be controlled by applying the gate voltage. Although the role of spin-orbital and magnetic (Zeeman) fields in the electric and spin transport is well understood for a noninteracting quantum wire,[3–7] the case of interacting electrons remains the subject of active research.[8–16]

It should be noted that finite spin-orbit coupling is very natural, and strictly speaking, unavoidable, in semiconducting quantum wires due to pronounced structural asymmetry inherent in the fabrication process. Also, in addition to the noted asymmetry of confining potentials (which include quantum-well potential that confines electrons to a two-dimensional (2D) layer as well as transverse (in-plane) potential that forms the one-dimensional channel[3]), spin-orbit interaction is inherent to semiconductors of either zinc-blende or wurtzite lattice structures lacking inversion symmetry.[17]

Another very interesting system that motivates our investigation is provided by one-dimensional electron surface states on vicinal surface of gold[18] as well as by electron states of self-assembled gold chains on stepped Si(111) surface of silicon.[19] In both of the systems, one-dimensional ballistic channels appear due to atomic reconstruction of surface layer of atoms, see also Refs. 20 and 21. The resultant surface electronic states lie within the band gap of bulk states, and thus, to high accuracy, are decoupled from electrons in the bulk. Spin-orbit interaction is unexpectedly strong in these systems, with the spin-orbit energy splitting of the order of 100 meV. In fact, spin-split subbands of Rashba type have been observed in angular resolved photoemission spectroscopy (ARPES) in both two-dimensional[22] and one-dimensional settings.[18,19] The very fact that the two (horizontally) spin-split parabolas are observed in ARPES speaks for high quality and periodicity of the obtained surface channels.

As we show below, the most interesting situation involves electrons subjected to both spin-orbital and magnetic fields. While it is perhaps impractical to think of ARPES measurements in the presence of magnetic field, it is quite possible to imagine experiments on *magnetic* metal surfaces.[23,24] It is then natural to investigate combined effect of noncommuting spin-orbit and Zeeman interactions, together with electron-electron interaction, on the one-dimensional system of electrons.

Electrons in a quantum wire (or, in a one-dimensional surface channel) are a good realization for a Tomonaga-Luttinger liquid and serve as an ideal system for the study of the interplay of magnetic field and Rashba spin-orbit effect on the interacting quantum wire. The magnetic field breaks the time-reversal symmetry of the Hamiltonian and splits the band of free electrons into two, corresponding to up-spin and down-spin electrons, reducing spin-rotational symmetry of the system from SU(2) to U(1). Subsequent inclusion of the Rashba term, $H_R \propto \vec{\sigma}\times\vec{p}\cdot\hat{z}$, see Eq. (2), breaks this U(1) symmetry (observe that $[\sigma_z, H_R]\neq 0$). In addition, the spin-orbit interaction (SOI) H_R breaks spatial inversion symmetry $\mathcal{P}: x\to -x$. A consequence of the fully broken SU(2) symmetry is the generation of new scattering processes which are no longer spin conserving. These are the Cooper scattering processes in which a pair of electrons in the lower band scatter to the upper band and vice versa.[25] A relevant Cooper term creates a gap in the energy spectrum leading to a long-range spin-density wave order. We find that the gap strength is proportional to the backscattering ($2k_F$ component) of the electron-electron interaction potential and the ratio of the Rashba to the Zeeman energy. For a large enough gap the ordering in the spin-density wave can crucially suppress the backscattering process of electrons from an isolated impurity. Brief description of our main results was previously given in Ref. 25.

We will also analyze an alternate system, a Mott-Hubbard Heisenberg spin-1/2 chain, in the presence of magnetic field and Dzyloshinskii-Moriya (DM) interaction. A magnetic field applied at an angle perpendicular to the DM anisotropy axis breaks the continuous U(1) symmetry and consequently a true long-range order can develop in the spin chain. The case of a staggered DM term has been studied experimentally[26] and theoretically,[27] and has been shown to open up a gap in the spectrum with the gap scaling as $B^{2/3}$ with the magnetic field B. The case of a uniform DM term is also experimentally relevant, see for example Ref. 28, but has not been discussed much theoretically. We will show that the case of a uniform DM term and perpendicular magnetic field can be described analogously to the quantum wire in the presence of spin-orbit interaction and magnetic field.

The outline for the paper is as follows: In Sec. II we will review noninteracting electrons in the presence of magnetic field and Rashba spin-orbit term. We then consider interaction effects by using standard bosonization approach. In Sec. III B we will perform renormalization-group analysis to determine relevant and irrelevant terms. In Sec. IV we use perturbative approach to generate relevant terms. The role of relevant terms on the transport property of electrons in the presence of a single impurity is analyzed in Sec. V. In Sec. VI, we consider a spin-1/2 Heisenberg antiferromagnetic chain in the presence of magnetic field and uniform DM term and compare this system with the quantum wire. Comparison with the case of a staggered DM term is considered in Sec. VI D. In Sec. VII we discuss the role ESR measurements can play in unraveling the DM term in the spin chain. Technical details of our calculations are described in Appendices.

II. ONE-DIMENSIONAL ELECTRONS IN THE PRESENCE OF MAGNETIC FIELD AND RASHBA SPIN ORBIT TERM

The Hamiltonian for an electron subject to the Rashba spin-orbit term and in the presence of magnetic field is given by

$$H_0 = \frac{\hbar^2(p_x^2+p_y^2)}{2m} + V(x) - g\mu_B\frac{\vec{\sigma}}{2}\cdot\vec{B} + H_R, \quad (1)$$

$$H_R = \frac{\alpha_R}{\hbar}(\sigma_x p_y - \sigma_y p_x), \quad (2)$$

where α_R is the Rashba spin-obrit coupling, g is the effective Bohr magneton, B is the magnetic field, σ_μ ($\mu = x,y,z$) are the Pauli matrices and the potential $V(x)=m\omega^2x^2/2$ typically confines the particle in the x direction. When the confining potential is strong enough so that the width of the wire $\sqrt{\hbar/(2m\omega)}$ is much smaller than the electron Fermi wavelength, only the first subband is occupied and the Hamiltonian (1) acquires a one-dimensional form

$$H_0 \approx \frac{\hbar^2k^2}{2m} + \frac{\alpha_R}{\hbar}(\sigma_x k) - g\mu_B\frac{\vec{\sigma}}{2}\cdot\vec{B}. \quad (3)$$

Here and below k is electron's momentum along the axis of the wire, which we will denote as x axis in the following for notational convenience. We also set $\hbar=1$.

It is easy to see that in the *absence* of magnetic-field SOI in Eq. (3) can be easily gauged away via the *spin-dependent* shift of the momentum, $H_0(B=0)\propto(k+m\alpha_R\sigma_x)^2$. Corrections arising from the omitted term $\alpha_R\sigma_y p_x$, Eq. (2), produce small spin-dependent variations of the velocities of right- and left-moving particles.[10] These, however, are not important for our purposes for as long as $\alpha_R k_F \ll E_F$, which is the limit (along with $\Delta_Z \ll E_F$) considered in this work. With interactions included, electrons form Luttinger liquid with somewhat modified critical exponents, in comparison with the standard case of no SOI.[10,11]

Most interesting situation arises when both SOI and Zeeman terms are present simultaneously and do not commute with each other as happens when spin-orbital axis [σ_x in (3)] is different from the magnetic-field direction. In what follows we choose magnetic field to point along the z direction, $\vec{B}=B\hat{z}$. The energy eigenvalues of the Hamiltonian (3) is found as[5,6]

$$\epsilon_\pm = \frac{k^2}{2m} \pm \sqrt{(\alpha_R k)^2 + \left(\frac{\Delta_z}{2}\right)^2}, \quad (4)$$

and the momentum dependent eigenspinors are, for $\epsilon_+(k)$:

$$|\chi_+(k)\rangle \equiv \begin{bmatrix} \sin\frac{\gamma(k)}{2} \\ \cos\frac{\gamma(k)}{2} \end{bmatrix}, \quad (5)$$

and for $\epsilon_-(k)$:

$$|\chi_-(k)\rangle \equiv \begin{bmatrix} \cos\frac{\gamma(k)}{2} \\ -\sin\frac{\gamma(k)}{2} \end{bmatrix}, \quad (6)$$

where $\Delta_z = g\mu_B B$ and rotation angle $\gamma(k)$ is introduced

$$\gamma(k) = \arctan\frac{2\alpha_R k}{\Delta_z}. \quad (7)$$

Notice that particles with momentum $\pm k$ experience an effective magnetic field $\vec{B}_{\rm eff} = \mp 2\alpha_R k/(g\mu_B)\hat{x} + B\hat{z}$, thus spin directions in each band vary with the momentum. Going from left to the right side of the band, the spins "rotate" along the counter-clockwise direction. At $k=0$, the effective magnetic field is just the applied field, thus the separation between the two bands is minimum and the spins align according to the applied field, i.e., along the $\mp z$ direction. For states with the same energy the spin states of + and − bands are no longer orthogonal if there is a finite magnetic field and Rashba spin-orbit coupling. In particular the right and left Fermi levels satisfy the following property

$$k_-^{R/L} + k_+^{R/L} \approx \pm 2k_F \quad (8)$$

and

$$\delta k_F = |k_-^{R/L} - k_+^{R/L}| \approx \frac{m}{k_F}\sqrt{(2k_F\alpha_R)^2 + (\Delta_z)^2}. \quad (9)$$

The magnitude of velocities at the Fermi level are

$$|u_\pm| \approx v_F \mp \frac{(\Delta_z)^2}{2k_F\sqrt{4(\alpha_R k_F)^2 + (\Delta_z)^2}}, \tag{10}$$

where $v_F = k_F/m$. The spin overlap between the upper (+) and lower (−) band is nonzero,

$$\langle \chi(k_+^{R/L}) | \chi(k_-^{R/L}) \rangle = \sin\frac{\gamma(k_+^{R/L}) - \gamma(k_-^{R/L})}{2}. \tag{11}$$

As will be discussed in Sec. III, the nonorthogonality of the spin states acquires important consequences when one turns on the electron-electron interactions.

III. INTERSUBBAND INTERACTION EFFECTS

The interaction part in terms of the particle-field operators $\Psi_\sigma(x)$ and $\Psi_\sigma^\dagger(x)$ (σ and σ' are the spin indices) is given by

$$H_{\rm int} = \frac{1}{2}\int dx dx' \, U(x-x')\Psi_\sigma^\dagger(x)\Psi_{\sigma'}^\dagger(x')\Psi_{\sigma'}(x')\Psi_\sigma(x), \tag{12}$$

where the summation on pairs of identical spin indices is assumed and $U(x-x')$ is the screened (by surrounding gates) interaction between the electrons. The field $\Psi_\sigma(x)$ is conventionally defined in terms of the annihilation operator $a_\sigma(k)$ of a free particle in the state e^{ikx} and with spin $\sigma = \uparrow, \downarrow$: $\Psi_\sigma(x) = \int \frac{dk}{2\pi} e^{ikx} a_\sigma(k)$. Alternatively, annihilation operators $a_\pm(k)$ of particles, which are the eigenstates of the Hamiltonian (1) with eigenenergies $\epsilon_\pm(k)$, can be used to represent the field operator as follows:

$$\Psi_\sigma(x) = \sum_{\nu=\pm} \int \frac{dk}{2\pi} e^{ikx} \langle \chi_\nu(k) | \sigma \rangle a_\nu(k). \tag{13}$$

The low-energy physics of the interacting wire is described by linearizing the spectrum near the Fermi points, $\pm k_\pm$. The $\sigma = \uparrow, \downarrow$ field operators are now described, in coordinate space, in terms of the chiral right (R_ν) and left (L_ν) movers of $\nu = \mp$ subbands as follows

$$\Psi_\sigma(x) = \sum_{\nu=\mp} \langle \chi_\nu(k_\nu) | \sigma \rangle e^{ik_\nu x} R_\nu + \langle \chi_\nu(-k_\nu) | \sigma \rangle e^{-ik_\nu x} L_\nu. \tag{14}$$

Following,[29] we decompose the interaction part of Hamiltonian (12) into intrasubband $H_{\rm intra}$ and intersubband $H_{\rm inter}$ scattering processes. The intrasubband process, $H_{\rm intra}$, describes the interaction between electrons in the same subband and involves the standard forward and backscattering processes, see Appendix A. The second scattering mechanism, $H_{\rm inter}$, involves scattering between electrons in different subbands and can be conveniently divided into *forward*, *backward,* and *Cooper* scattering processes. Below we will discuss the intersubband scattering processes in more detail. The forward-scattering process involves interaction between $q \approx 0$ components of the densities in the two subbands

$$H_{\rm inter}^F = \frac{1}{2}U(0)\int dx \sum_{\nu=\pm} (R_\nu^\dagger R_\nu + L_\nu^\dagger L_\nu) \times (R_{-\nu}^\dagger R_{-\nu} + L_{-\nu}^\dagger L_{-\nu}). \tag{15}$$

The intersubband backscattering process is classified into *direct* and *exchange* scattering. Direct backscattering process involves $q \approx 2k_\pm$ components of the densities in the two subbands: a left (right) moving electron in the subband ν ($-\nu$) changes its direction to become a right (left) moving one while remaining in the same band,

$$H_{\rm inter}^{d-B} = \cos[\gamma(k_+)]\cos[\gamma(k_-)]U(k_+ + k_-) \times \sum_{\nu=\pm} \int dx e^{i2(k_\nu - k_{-\nu})x} (R_\nu^\dagger L_\nu)(L_{-\nu}^\dagger R_{-\nu}). \tag{16}$$

This contribution involves an oscillatory factor $\exp[i2(k_\nu - k_{-\nu})x]$ in the integral due to the nonconservation of momentum during the scattering.

The other backscattering process is via a momentum-conserving *exchange* mechanism, where electrons again scatter by large-momentum transfer and in the process *exchange their bands*. The scattering channel conserves momentum and reads

$$H_{\rm inter}^{\rm ex-B} = \frac{1}{2}\sum_{\nu=\pm}\left\{ U(k_\nu - k_{-\nu})\sin^2\left[\frac{\gamma(k_\nu) - \gamma(k_{-\nu})}{2}\right] \times \int dx [(R_\nu^\dagger R_{-\nu})(R_{-\nu}^\dagger R_\nu) + (L_\nu^\dagger L_{-\nu})(L_{-\nu}^\dagger L_\nu)] + 2U(k_\nu + k_{-\nu})\sin^2\left[\frac{\gamma(k_\nu) + \gamma(k_{-\nu})}{2}\right] \times \int dx (R_\nu^\dagger L_{-\nu})(L_{-\nu}^\dagger R_\nu)\right\}. \tag{17}$$

Note the appearance of (squared) wave-function overlap factors, $\propto \langle \chi(k_\nu) | \chi(\pm k_{-\nu}) \rangle$, which signify the exchange nature of the scattering. Note also that these factors are nonzero due to a finite Rasbha coupling α_R, which allows electrons to scatter without conserving their spins.

The Cooper scattering process, which is central to our story, involves scattering of a pair of opposite movers (right and left) in the subband ν into a similar pair in the other, $-\nu$, subband. Each pair has zero total momentum which remains conserved in this scattering. Being *pair-tunneling*-like, Cooper scattering requires *nonconservation* of spin. It represents, for example, a scattering of a pair of two (almost) down-spin electrons into a pair of two (almost) up-spin ones. The Cooper scattering reads

$$H_{\rm inter}^C = \int dx \left\{ U(k_- - k_+)\sin^2\left[\frac{\gamma(k_-) - \gamma(k_+)}{2}\right] - U(k_- + k_+)\sin^2\left[\frac{\gamma(k_-) + \gamma(k_+)}{2}\right]\right\} \times (R_-^\dagger L_-^\dagger R_+ L_+ + \text{h.c.}). \tag{18}$$

The first term (*direct* Cooper scattering) is due to electrons in the band R_ν and L_ν jumping into $R_{-\nu}$ and $L_{-\nu}$, respectively. The coefficient for this term is $\propto U(\delta k_F)\sin^2[(\gamma_- - \gamma_+)/2]$, where $\delta k_F = |k_\nu - k_{-\nu}|$ is the momentum transfer for an electron and $\sin^2[(\gamma_- - \gamma_+)/2]$ is the squared overlap integral. (For brevity, we denote $\gamma(k_\pm) = \gamma_\pm$ here and in the follow-

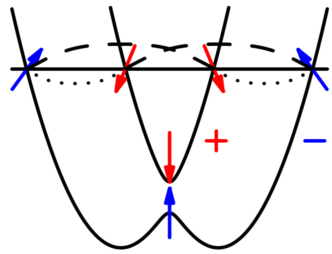

FIG. 1. (Color online) Occupied subbands $\epsilon_{\pm}$ of Eq. (4). Arrows illustrate spin polarization in different subbands. Dashed (dotted) lines indicate exchange (direct) Cooper scattering processes.

ing.) The second process (*exchange*) with electron scattering from R_ν and L_ν to $L_{-\nu}$ and $R_{-\nu}$, respectively, involves a coefficient, $\propto U(k_\nu+k_{-\nu})\sin^2[(\gamma_-+\gamma_+)/2]$, with a larger, $\sin^2[(\gamma_-+\gamma_+)/2]$, overlap integral. The bigger overlap for this second ($R_\nu \leftrightarrow L_{-\nu}$) process is also rather clear from pictorial representation of spin orientation in different subbands, as shown in Fig. 1. For the case of short-ranged (screened) interaction potential a simple estimate, using Eqs. (8) and (9),

$$\frac{U(k_- - k_+)\sin^2\left[\frac{\gamma_- - \gamma_+}{2}\right]}{U(k_- + k_+)\sin^2\left[\frac{\gamma_- + \gamma_+}{2}\right]} \approx \frac{U(\delta k_F)}{U(2k_F)}\left(\frac{\Delta_z}{2E_F}\right)^2 \ll 1 \qquad (19)$$

shows that the second, *exchange* Cooper process, dominates. This defines the regime to be considered in this work.

Finally we take into account two classes of momentum nonconserving scattering processes where one of them exhibits mixed features of Cooper and back-scattering (called as asymmetric back-scattering process) while the other one has features reminiscent of Cooper and forward-scattering processes (asymmetric forward-scattering process). A typical asymmetric back (forward) scattering event involves total momentum change $\pm\delta k_F$, see Fig. 2. For example, right and left moving fermions in the same subband scatter into left (right) and right (left) moving fermions, respectively, with one of the fermions now in a different subband. Alternatively, the oppositely moving fermions may be in different subbands to begin with but end up in the same subband with opposite (same) momentums. The scattering process acquires a slowly {compared to direct back scattering [Eq. (16)]} oscillating factor $\exp[i(k_\nu - k_{-\nu})x]$ owing to the nonconservation of momentum and is given by

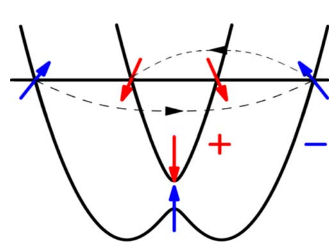

FIG. 2. (Color online) Asymmetric back-scattering processes.

$$H_{\text{asymm}} = \sin\left(\frac{\gamma_+ + \gamma_-}{2}\right)\sum_{\nu=\pm}\text{sgn}(\nu)U(2k_F)\cos(\gamma_\nu) \times \int dx[e^{i(k_\nu - k_{-\nu})x}(R_\nu^\dagger L_\nu^\dagger R_\nu L_{-\nu} - R_{-\nu}^\dagger L_\nu^\dagger R_\nu L_\nu) + \text{h.c.}]. \qquad (20)$$

The above expression reflects contributions from only the asymmetric back-scattering processes. The asymmetric forward-scattering processes involve identical fermion operators as in Eq. (20). However, the ratio of amplitudes for asymmetric forward to asymmetric back-scattering process is small,

$$\frac{U(\delta k_F)\sin\left(\frac{\gamma_\nu - \gamma_{-\nu}}{2}\right)}{U(2k_F)\sin\left(\frac{\gamma_\nu + \gamma_{-\nu}}{2}\right)\cos\,\gamma_\nu} \approx \frac{U(\delta k_F)}{U(2k_F)}\frac{\sqrt{\Delta_z^2 + 4(\alpha_R k_F)^2}}{2E_F} \ll 1,$$

which allows us to neglect contributions from asymmetric forward-scattering processes altogether.

The electron density in the quantum wire is assumed to be incommensurate with the lattice spacing, hence the Umklapp scattering process is not considered. To summarize, the interaction part of the Hamiltonian has been decomposed in terms of three broadly defined scattering processes, intrasubband, intersubband, and asymmetric scattering process,

$$H = H_{\text{intra}} + H_{\text{inter}} + H_{\text{Asymm}}. \qquad (21)$$

A. Bosonization

Bosonization is performed by expressing the fermionic operators in the Hamiltonian via the chiral bosonic ϕ_{R/L_ν} fields.[30–32] The fermionic fields in terms of the chiral bosonic field are as follows

$$R_\pm = \frac{\eta_\pm}{\sqrt{2\pi a_0}}e^{i\sqrt{4\pi}\phi_{R\pm}}, \quad L_\pm = \frac{\eta_\pm}{\sqrt{2\pi a_0}}e^{-i\sqrt{4\pi}\phi_{L\pm}}, \qquad (22)$$

where $a_0 \sim k_F^{-1}$ is the short-distance cutoff and $\eta_\pm$ are the Klein factors which are introduced to ensure the correct anticommutation relations for the fermionic operators from different ($\pm$) subbands. The bosonic operators obey the following commutation relations:

$$[\phi_{R\nu}, \phi_{L\nu'}] = \frac{i}{4}\delta_{\nu\nu'}; \quad \text{where} \quad \nu, \nu' = \pm \qquad (23)$$

$$[\phi_{R/L\nu}(x), \phi_{R/L\nu'}(y)] = \pm\frac{i}{4}\delta_{\nu\nu'}\,\text{sign}(x-y), \qquad (24)$$

the first of which, Eq. (23), ensures anticommutation between right and left movers from the *same* subband, while the second is needed for the anticommutation between like species (i.e., right with right, left with left). Klein factors anticommute

$$\{\eta_\nu, \eta_{\nu'}\} = 2\delta_{\nu\nu'}, \quad \eta_\nu^\dagger = \eta_\nu. \qquad (25)$$

In the following we choose the gauge where $\eta_+\eta_- = i$. The chiral $\phi_{R/L\nu}$ are expressed in terms of ϕ_ν and its dual θ_ν as follows

$$\phi_{R\nu} = \frac{\phi_\nu - \theta_\nu}{2}; \quad \phi_{L\nu} = \frac{\phi_\nu + \theta_\nu}{2}, \tag{26}$$

The bosonized form of the Hamiltonian is obtained by making use of Eqs. (22)–(26), as well as the following results for (chiral) densities

$$R_\nu^\dagger R_\nu = \frac{\partial_x \phi_{R\nu}}{\sqrt{\pi}} = \frac{\partial_x(\phi_\nu - \theta_\nu)}{\sqrt{4\pi}},$$

$$L_\nu^\dagger L_\nu = \frac{\partial_x \phi_{L\nu}}{\sqrt{\pi}} = \frac{\partial_x(\phi_\nu + \theta_\nu)}{\sqrt{4\pi}}. \tag{27}$$

The (bosonized) Hamiltonian in terms of ϕ_ν and θ_ν is the sum of intrasubband, intersubband scattering and asymmetric scattering processes,

$$H = H_{\text{intra}} + H_{\text{inter}} + H_{\text{asymm}}, \tag{28}$$

where the intrasubband part has the usual form (see Appendix A)

$$H_{\text{intra}} = \frac{1}{2}\sum_{\nu=\pm}\int dx\Bigg\{ v_F(\partial_x\theta_\nu)^2 + \left[v_F + \frac{U(0) - \cos^2[\gamma(k_\nu)]U(2k_\nu)}{\pi}\right](\partial_x\phi_\nu)^2\Bigg\}. \tag{29}$$

The intersubband part of the Hamiltonian, H_{inter}, is given by

$$H_{\text{inter}} = H_{\text{inter}}^F + H_{\text{inter}}^{d-B} + H_{\text{inter}}^{\text{ex}-B} + H_{\text{inter}}^C, \tag{30}$$

where

$$H_{\text{inter}}^F = \frac{U(0)}{\pi}\int dx \partial_x\phi_+\partial_x\phi_-,$$

$$H_{\text{inter}}^{d-B} = \frac{U(2k_F)\cos^2[\gamma_F]}{2(\pi a_0)^2}\int dx \cos[2\sqrt{\pi}(\phi_- - \phi_+) + 2(k_+ - k_-)x],$$

$$H_{\text{inter}}^{\text{ex}-B} = -\frac{U(2k_F)}{2\pi}\sin^2[\gamma_F]\int dx(\partial_x\phi_+\partial_x\phi_- - \partial_x\theta_+\partial_x\theta_-),$$

$$H_{\text{inter}}^C = \frac{U(2k_F)\sin^2[\gamma_F]}{2(\pi a_0)^2}\int dx \cos[2\sqrt{\pi}(\theta_- - \theta_+)]. \tag{31}$$

The asymmetric part has the following bosonized form,

$$H_{\text{asymm}} = -\frac{\sqrt{2}U(2k_F)\sin(2\gamma_F)}{(2\pi)^{3/2}a_0}\int dx\{\partial_x(\phi_{R-} - \phi_{R+}) \times \sin[\sqrt{4\pi}(\phi_{L-} - \phi_{L+}) - \delta k_F x] + \partial_x(\phi_{L-} - \phi_{L+}) \times \sin[\sqrt{4\pi}(\phi_{R-} - \phi_{R+}) - \delta k_F x]\}. \tag{32}$$

In deriving Eqs. (31) and (32) we took limits $\Delta_z \ll E_F = v_F k_F$ and $\alpha_R k_F \ll E_F$ which allowed us to neglect velocity differences [Eq. (10)] in the two subbands and approximate $U(2k_\pm)$, $U(\frac{3k_\nu - k_{-\nu}}{2}) \approx U(2k_F)$ and $\gamma(k_\pm) \approx \gamma(k_F) \equiv \gamma_F$. An important exception to this replacement is provided by H_{inter}^{d-B} and H_{asymm} in Eqs. (31) and (32), respectively, where momentum *mismatch* factors $2(k_- - k_+)x = 2\delta k_F x$ and $(k_- - k_+)x = \delta k_F x$ must be preserved. It is worth noting here that the approximations assumed do not restrict the ratio $2\alpha_R k_F/(g\mu_B B) = E_{s-o}/\Delta_z$, which can still take on any value.

A more standard representation of the Hamiltonian is in terms of the symmetric ϕ_ρ, θ_ρ (*charge*) and antisymmetric ϕ_σ, θ_σ (*spin*) modes. These combinations are defined as follows

$$\varphi_\rho = \frac{\phi_- + \phi_+}{\sqrt{2}}, \quad \varphi_\sigma = \frac{\phi_- - \phi_+}{\sqrt{2}},$$

$$\theta_\rho = \frac{\theta_- + \theta_+}{\sqrt{2}}, \quad \theta_\sigma = \frac{\theta_- - \theta_+}{\sqrt{2}}. \tag{33}$$

The Hamiltonian now reads

$$H = H_\rho + H_\sigma. \tag{34}$$

The charge part of the Hamiltonian is harmonic

$$H_\rho = \frac{1}{2}\int dx\left[u_\rho K_\rho(\partial_x\theta_\rho)^2 + \frac{u_\rho}{K_\rho}(\partial_x\varphi_\rho)^2\right]. \tag{35}$$

The spin part is the sum of quadratic and nonlinear terms, $H_\sigma = H_\sigma^0 + H_\sigma^C + H_\sigma^B + H_\sigma^A$, where

$$H_\sigma^0 = \frac{1}{2}\int dx\left[u_\sigma K_\sigma(\partial_x\theta_\sigma)^2 + \frac{u_\sigma}{K_\sigma}(\partial_x\varphi_\sigma)^2\right] \tag{36}$$

$$H_\sigma^C = \frac{U(2k_F)\sin^2[\gamma_F]}{2(\pi a_0)^2}\int dx \cos[\sqrt{8\pi}\theta_\sigma] \tag{37}$$

$$H_\sigma^B = \frac{U(2k_F)\cos^2[\gamma_F]}{2(\pi a_0)^2}\int dx \cos[\sqrt{8\pi}\varphi_\sigma + 2\delta k_F x] \tag{38}$$

$$H_\sigma^A = H_{\text{asymm}} = i\eta_+\eta_-\frac{U(2k_F)\sin(2\gamma_F)}{(2\pi)^{3/2}a_0}\int dx\{\partial_x(\phi_\sigma - \theta_\sigma)\sin[\sqrt{2\pi}(\phi_\sigma + \theta_\sigma) - \delta k_F x] + \partial_x(\phi_\sigma + \theta_\sigma)\sin[\sqrt{2\pi}(\phi_\sigma - \theta_\sigma) - \delta k_F x]\}. \tag{39}$$

For completeness, it is worth noting that the leading correction to these equations is represented by the *intermode* term

$$H'_{\rho-\sigma} = \frac{(u_+ - u_-)}{2}\int dx(\partial_x\varphi_\rho\partial_x\varphi_\sigma + \partial_x\theta_\rho\partial_x\theta_\sigma), \tag{40}$$

which couples spin and charge sectors. Its small amplitude $(u_+ - u_-) \propto \Delta_z/E_F$, see Eq. (10), justifies its neglect in the following.

Competing nature of interacting problem is clear from the presence of two nonlinear terms, Eqs. (37) and (38), involving noncommuting (dual) boson fields θ_σ and φ_σ, in the

Hamiltonian. Similar situation happens in models of organic conductors, where spin-nonconserving spin-orbit and dipole-dipole interactions play an important role.[33]

The Luttinger liquid parameters, $K_{\rho/\sigma}$, and charge/spin velocities $u_{\rho/\sigma}$ are found by adding contributions from harmonic Hamiltonians (29) and H^F_{inter} and $H^{\text{ex}-B}_{\text{inter}}$ from Eq. (31), with the result

$$u_\rho \approx v_F\left[1 + \frac{2U(0) - U(2k_F)\cos^2(\gamma_F)}{2\pi v_F}\right],$$

$$u_\sigma \approx v_F\left[1 - \frac{U(2k_F)\cos^2(\gamma_F)}{2\pi v_F}\right],$$

$$K_\rho \approx 1 - \frac{2U(0) - U(2k_F)}{2\pi v_F} \leq 1,$$

$$K_\sigma \approx 1 + \frac{U(2k_F)\cos(2\gamma_F)}{2\pi v_F}. \tag{41}$$

These expressions are perturbative in small parameters $U(0)/v_F$ and $U(2k_F)/v_F$. Note also that physically reasonable interactions are characterized by $U(2k_F) \leq U(0)$, where the equality sign is obtained in the limit of fully screened, delta-function-like contact interaction between electrons.

Noting that γ_F varies from 0 to $\pi/2$ as the ratio E_{s-o}/Δ_z varies from 0 to ∞,

$$\gamma_F = \arctan\frac{2\alpha_R k_F}{\Delta_z} \to \begin{cases} 0 & \text{for } E_{s-o} \ll \Delta_z \\ \pi/2 & \text{for } E_{s-o} \gg \Delta_z \end{cases}, \tag{42}$$

we observe that spin stiffness K_σ in Eq. (41) varies from its standard value slightly *above* 1, $K_\sigma(\gamma_F \to 0) = 1 + U(2k_F)/(2\pi v_F)$, to the value *below* 1, $K_\sigma(\gamma_F \to \pi/2) = 1 - U(2k_F)/(2\pi v_F)$. This unusual behavior, consequences of which are discussed below, is rooted in the spin-orbit-broken spin-rotational invariance of the problem, as discussed in Sec. I.

B. Renormalization group analysis

The fate of the three nonlinear terms, Cooper [Eq. (37)], backscattering [Eq. (38)], and asymmetric [Eq. (39)], are determined by renormalization-group (RG) analysis. The analysis is significantly simplified by expressing the Hamiltonian in terms of current operators. To this end, we write the Hamiltonian in terms of the right and left spin $(\vec{J}_R, \vec{J}_L)$ and charge (J^c_R, J^c_L) currents, which obey Kac-Moody algebra.[30] The uniform part of the currents is expressed in terms of the chiral right and left moving fermions: the charge currents are

$$J^c_R = \sum_{\nu=\mp} R^\dagger_\nu R_\nu, \quad J^c_L = \sum_{\nu=\mp} L^\dagger_\nu L_\nu, \tag{43}$$

and the spin currents are

$$\vec{J}_R = \sum_{\nu\nu'=\mp} R^\dagger_\nu \frac{\vec{\sigma}_{\nu\nu'}}{2} R_{\nu'}, \quad \vec{J}_L = \sum_{\nu\nu'=\mp} L^\dagger_\nu \frac{\vec{\sigma}_{\nu\nu'}}{2} L_{\nu'}. \tag{44}$$

As an example, the z component of the right moving spin current is defined as $J^z_R = (R^\dagger_- R_- - R^\dagger_+ R_+)/2$. Note that in the asymptotic limit of $\alpha_R k_F/\Delta_Z \to 0$ the $(-,+)$ bands correspond to $(\uparrow,\downarrow)$ spin bands and we recover the canonical definition for the spin current.

Since the charge part of the Hamiltonian is quadratic, Eq. (35), and is decoupled from the spin part, it suffices to consider the RG flow of the spin part only, $H_\sigma = H^0_\sigma + H^C_\sigma + H^B_\sigma + H^A_\sigma$. In terms of current operators it reads:

$$H^0_\sigma = 2\pi u_\sigma \int dx[(J^z_R J^z_R + J^z_L J^z_L) - y_\sigma J^z_R J^z_L],$$

$$H^A_\sigma = \pi u_\sigma y^A_\sigma \int dx[e^{-i\delta k_F x}(J^z_R J^-_L - J^+_R J^z_L) + \text{h.c.}],$$

$$H^B_\sigma = \pi u_\sigma y^B_\sigma \int dx(e^{-i2\delta k_F x} J^-_R J^+_L + \text{h.c.}),$$

$$H^C_\sigma = \pi u_\sigma y^C_\sigma \int dx(J^-_R J^-_L + \text{h.c.}). \tag{45}$$

The initial values $(\ell = 0)$ of the interaction parameters are

$$y_\sigma(0) = 2(K_\sigma - 1) = U(2k_F)\cos(2\gamma_F)/\pi v_F,$$

$$y^C_\sigma(0) = U(2k_F)\sin^2(\gamma_F)/\pi u_\sigma,$$

$$y^B_\sigma(0) = -U(2k_F)\cos^2(\gamma_F)/\pi u_\sigma,$$

$$y^A_\sigma(0) = U(2k_F)\sin(2\gamma_F)/\pi u_\sigma. \tag{46}$$

Note that to first order in $U(2k_F)$ there is no difference between v_F and u_σ in denominators of the above expressions.

It is convenient to start with formal but useful limit of $\delta k_F = 0$, where H_σ can be compactly written in terms of spin currents

$$H_\sigma(\delta k_F = 0) = 2\pi u_\sigma \int dx\Bigg[(J^z_R J^z_R + J^z_L J^z_L) + \sum_{a=x,y,z} y_a J^a_R J^a_R + y_A(J^z_R J^x_L - J^x_R J^z_L)\Bigg]. \tag{47}$$

Here $y_x = y^B_\sigma + y^C_\sigma$, $y_y = y^B_\sigma - y^C_\sigma$, $y_z = -y_\sigma$, and $y_A = y^A_\sigma$. The RG equations for the dimensionless couplings $y_{a=x,y,z,A}$ are easy to derive with the help of operator product expansion (OPE) technique,

$$\frac{dy_x}{d\ell} = y_y y_z,$$

$$\frac{dy_y}{d\ell} = y_z y_x + y_A^2,$$

$$\frac{dy_z}{d\ell} = y_y y_x,$$

$$\frac{dy_A}{d\ell} = y_y y_A. \tag{48}$$

Despite complicated appearance, the solution of this system of equations is easy. One finds that $y_x(\ell)=y_z(\ell)$, $y_A(\ell)=-\sin(2\gamma_F)y_y(\ell)$, and $y_x(\ell)=\cos(2\gamma_F)y_y(\ell)$. As a result, the system is reduced to a single equation $dy_y/d\ell=y_y^2$, solution of which is standard: $y_y(\ell)=y_y(0)/(1-y_y(0)\ell)\to -1/\ell$ for $\ell\to\infty$.

Thus, in the absence of momentum mismatch δk_F between the two subbands, *all* perturbations in Eq. (47) are marginally irrelevant and logarithmically decay to zero. This simply reflects rotational $SU(2)$ symmetry of the problem in the absence of spin-orbit and Zeeman fields.

The full problem, with $\delta k_F \neq 0$, is solved by neglecting all momentum nonconserving terms in H_σ: being marginal in the $\delta k_F=0$ limit, such terms become infinitely irrelevant for $\delta k_F\neq 0$. More carefully, we can follow full RG [Eq. (48)] until $\ell_z=\ln[1/(a_0\delta k_F)]=\ln[k_F/\delta k_F]$ is reached—beyond this scale oscillating terms average to zero. The end result is that we are allowed to disregard $H_\sigma^{A,B}$ terms in Eq. (45). RG equations for remaining couplings can be obtained from Eq. (48) by nullifying all but two, y_σ and $y_\sigma^C\equiv y_C$, couplings. This leads to the standard system of two KT equations (see, for example, Refs. 30 and 31).

$$\frac{dy_\sigma}{d\ell} = y_C^2, \quad \frac{dy_C}{d\ell} = y_\sigma y_C. \tag{49}$$

Note that initial values of these couplings are given by corresponding solutions of Eq. (48) evaluated at $\ell=\ell_z$.

Solution of this system is determined by the integral of motion $\mu^2=y_C^2-y_\sigma^2$ and the ratio of the initial couplings $y_\sigma(0)/y_C(0)=-\cos(\alpha)$. In terms of these parameters it reads[30]

$$y_\sigma(\ell) = -\mu\cot(\mu\ell+\alpha), \quad y_C(\ell)=\mu/\sin(\mu\ell+\alpha). \tag{50}$$

There are three different regimes, illustrated in Fig. 3.

(I) strong coupling: $1/\sqrt{3}\geq \sin(\gamma_F)\geq 0$. Here $\mu=-im$, with $m>0$, and $\alpha=\pi+i\beta$, with $\beta>0$. In this regime both coupling flow to strong coupling, reaching pole singularity at $\ell_0=\beta/m$.

(II) cross-over regime: $1\geq\sin(\gamma_F)\geq 1/\sqrt{3}$. Here $\mu>0$ and $0\leq\alpha\leq\pi$. The flow is still to strong coupling, but via an intermediate (cross-over) region (for $0\leq\alpha\leq\pi/2$) where $y_C(\ell)$ initially decreases. Eventually both $y_{\sigma,C}$ reach strong coupling, at $\ell_0=(\pi-\alpha)/\mu$.

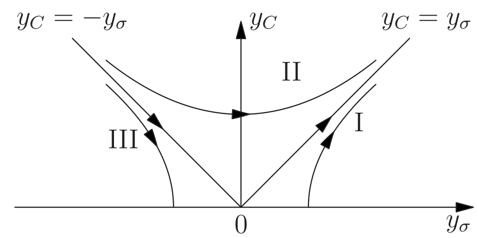

FIG. 3. RG flow of Eq. (49).

(III) weak coupling: This obtains when $\mu=im$, $m>0$, and $\alpha=i\beta$, with $\beta>0$. In this situation $y_\sigma\to -m$ as $\ell\to\infty$, while $y_C\to 0$. This is critical (Luttinger liquid) phase of the spin sector. It is, however, *not* realized in our problem as the requirement $-y_\sigma(0)>y_C(0)>0$ is equivalent to $\sin[\gamma_F]>1$ which is clearly not possible.

The conclusion is then that Cooper phase is realized for arbitrary value of $\pi/2>\gamma_F\geq 0$, i.e., for arbitrary ratio of SO to Zeeman energies, $\tan(\gamma_F)=2\alpha_R k_F/\Delta_z$. This finding of the Cooper phase, which has the meaning of the spin-orbit stabilized *spin-density-wave* (SDW_x) phase (see Sec. IV), constitutes the main result of our work. We have previously discussed the limit of small γ_F, which is physically most transparent, in Ref. 25.

C. The nature of the Cooper ordering

We now consider the physical meaning of the Cooper instability. For simplicity we focus on the regime **I** of Sec. II. Being relevant, the Cooper term (37) grows in magnitude and reaches strong-coupling limit when $y_C(\ell_c)\sim 1$ while $K_\sigma\to 2$.[30] A positive value of g_C results in θ_σ field being pinned to one of the semiclassical minima $\theta_\sigma^{cl}=(m+\frac{1}{2})\sqrt{\pi/2}$ ($m\in Z$). The energy cost of (massive) fluctuations $\delta\theta_\sigma$ near these minima represents spin gap which can be estimated as

$$\Delta_c \approx \frac{v_F}{\xi} = v_F\left[\frac{\gamma_F^2 U(2k_F)}{\pi v_F}\right]^{K_\sigma/2(K_\sigma-1)}. \tag{51}$$

Here $\xi=a_0 e^{\ell_c}$ is the correlation length, $\xi\sim\{\pi v_F/[U(2k_F)\gamma_F^2]\}^{K_\sigma/[2(K_\sigma-1)]}$. Physical meaning of these minima follows from the analysis of spin correlations.

We start with spin density $S^a=\Psi_s^\dagger\sigma_{s,s'}^a\Psi_{s'}/2$, which is defined with respect to the standard spin basis, $s=\uparrow,\downarrow$. We focus on "$2k_F$" components of spin, where quotation marks are used to remind that large-momentum components of spin density include contributions from both $k_+ + k_- = 2k_F$ and $2k_\pm$ processes, see Fig. 1. We find (using the gauge $\eta_+\eta_-=i$)

$$\begin{pmatrix} S^x \\ S^y \\ S^z\end{pmatrix}_{2k_F} = -\frac{\cos(\sqrt{2\pi}\varphi_\rho+2k_F x)}{\pi a_0}\times\begin{bmatrix} -\sin(\sqrt{2\pi}\theta_\sigma) \\ \cos(\gamma_F)\cos(\sqrt{2\pi}\theta_\sigma)+\sin(\gamma_F)\cos(\sqrt{2\pi}\varphi_\sigma+\delta k_F x) \\ \sin(\sqrt{2\pi}\varphi_\sigma)\end{bmatrix} \to -\frac{\cos(\sqrt{2\pi}\varphi_\rho+2k_F x)}{\pi a_0}\begin{pmatrix}\pm 1\\ 0\\ 0\end{pmatrix}. \tag{52}$$

The last line of the above equation is somewhat symbolic, with zeros representing *exponentially* decaying correlations of the corresponding spin components, $S^{y,z}$. Here $\hat{z}$ component is disordered by strong quantum fluctuations of *dual* φ_σ field, as dictated by $[\varphi,\theta]$ commutation relation, see Eq. (68). The $\hat{y}$ component does not order because $\cos[\sqrt{2\pi}\theta_\sigma^{\rm cl}]=0$. Thus Cooper order found here in fact represents spin-density-wave (SDW$_x$) order at momentum $2k_F$ of the $\hat{x}$ component of spin density, as discussed previously in Ref. 25. Observe that S^x ordering is of *quasi-LRO* type as it involves free charge boson, φ_ρ. As a result, spin correlations do decay with time and distance, but very slowly $\langle S^x(x)S^x(0)\rangle \sim \cos(2k_F x)x^{-K_\rho}$.

The result Eq. (52) also hints a possibility of truly long-range-ordered spin correlations in the insulating state of the wire Heisenberg spin chain. There the charge field φ_ρ is pinned by the relevant two-particle Umklapp scattering,[31] which can be mimicked by setting $K_\rho\to 0$ in the spin-correlation function above. This is the essence of the result to be discussed in Sec. VI below.

Observe another interesting feature of Eq. (52): $S^y_{2k_F}$ has the appearance of rotated by angle γ_F component of the vector, whereas $S^x_{2k_F}$ and $S^z_{2k_F}$ remain unchanged. The question that arises is what does $S^y_{2k_F}$ rotate into? The full answer is provided by considering $2k_F$ component of the generalized *helicity* operators

$$h^a_{2k_F}(x) = -(i/2k_F)\sum_{s,s'=\uparrow,\downarrow}\Psi^\dagger_s\sigma^a_{s,s'}\hat{p}\Psi_{s'}|_{2k_F}$$
$$= (i/2)\sum_{s=\uparrow,\downarrow}(R^\dagger_s\sigma^a_{s,s'}L_{s'}e^{-i2k_Fx} - L^\dagger_s\sigma^a_{s,s'}R_{s'}e^{i2k_Fx}), \tag{53}$$

where $a=\{0,x,y,z\}$. Note that $h^0_{2k_F}(x)$ turns into well-known staggered dimerization operator $\epsilon(x)$ in the "spin chain limit" of the problem, when charge fluctuations disappear. In terms of the right and left moving fermions the staggered $(2k_F)$ dimerization is given by

$$\epsilon = h^0_{2k_F} = (i/2)\sum_{s=\uparrow,\downarrow}(R^\dagger_s L_s e^{-i2k_Fx} - L^\dagger_s R_s e^{i2k_Fx}). \tag{54}$$

Its bosonized form, in terms of ϕ_σ and θ_σ fields [Eq. (33)],

$$h^0_{2k_F} = \frac{\cos(\sqrt{2\pi}\varphi_\rho + 2k_Fx)}{\pi a_0}[\cos(\gamma_F)\cos(\sqrt{2\pi}\varphi_\sigma + \delta k_F x) - \sin(\gamma_F)\cos(\sqrt{2\pi}\theta_\sigma)]. \tag{55}$$

matches "rotated" $S^y_{2k_F}$ in Eq. (52) exactly.

Although similar looking, this operator is different from "$2k_F$" component of the density, described below in Eq. (60). That one has charge boson φ_ρ appearing under sine, see Eqs. (61) and (62), while both ϵ and $\vec{S}$ fields are proportional to the cosine of it (see also Ref. 41). The difference is important. The y–component of spin and h^0 operators in the original up and down-spin basis can be written in a rather compact form,

$$S^y_{2k_F} = \cos(\gamma_F)\tilde{S}^y_{2k_F} + \sin(\gamma_F)\tilde{h}^0_{2k_F}$$
$$h^0_{2k_F} = \cos(\gamma_F)\tilde{h}^0_{2k_F} - \sin(\gamma_F)\tilde{S}^y_{2k_F}, \tag{56}$$

where

$$\tilde{S}^y_{2k_F}(x) = \frac{1}{2}\sum_{\nu,\nu'=\mp}[R^\dagger_\nu\sigma^y_{\nu,\nu'}L_{\nu'}e^{-i(k_\nu+k_{\nu'})x} + L^\dagger_\nu\sigma^y_{\nu,\nu'}R_{\nu'}e^{-i(k_\nu+k_{\nu'})x}] \tag{57}$$

and

$$\tilde{h}^0_{2k_F}(x) = \frac{i}{2}\sum_{\nu=\mp}(R^\dagger_\nu L_\nu e^{-i2k_\nu x} - L^\dagger_\nu R_\nu e^{i2k_\nu x}), \tag{58}$$

are, respectively, the spin and h^0 operators in the $\mp$ basis. Thus, in the limit of $\gamma_F\to\pi/2$, the $2k_F$ component of spin along the y direction in one basis appears as the h^0 operator in the second basis and vice versa.

Of the remaining staggered operators, h^a $(a=x,y,z)$ only h^y is affected by rotation. The y component partially "rotates" into the $2k_F$ part of the density operator, $\tilde{\rho}_{2k_F}=\Sigma_{\nu=\mp}(R^\dagger_\nu L_\nu e^{-i2k_\nu x}+L^\dagger_\nu R_\nu e^{i2k_\nu x})$, via the following relation

$$h^y_{2k_F} = \cos(\gamma_F)\tilde{h}^y_{2k_F} - \sin(\gamma_F)\tilde{\rho}_{2k_F}/2. \tag{59}$$

On the other hand the $2k_F$ component of density operator in the original spin basis, $\rho_{2k_F}=\Sigma_{s=\uparrow,\downarrow}(R^\dagger_s L_s e^{-i2k_Fx}+L^\dagger_s R_s e^{i2k_Fx})$, rotates into the y component of the h-operator

$$\rho_{2k_F}/2 = \cos(\gamma_F)\tilde{\rho}_{2k_F}/2 + \sin(\gamma_F)\tilde{h}^y_{2k_F}. \tag{60}$$

As before, tilde's are used to denote operators in the $\mp$ basis. The bosonized forms for $\tilde{h}^y_{2k_F}$ and $\tilde{\rho}_{2k_F}$ are as follows:

$$\tilde{h}^y_{2k_F} = -\frac{1}{\pi a_0}\sin(\sqrt{2\pi}\phi_\rho + 2k_Fx)\cos(\sqrt{2\pi}\theta_\sigma) \tag{61}$$

and

$$\tilde{\rho}_{2k_F} = -\frac{2}{\pi a_0}\sin(\sqrt{2\pi}\phi_\rho + 2k_Fx)\cos(\sqrt{2\pi}\phi_\sigma + \delta k_F x). \tag{62}$$

Note that the charge content of h^y and density operator are the same but the spin parts are different.

The following relation may be helpful in revealing the origins of h^a and ϵ fields. In the case of Heisenberg chain the staggered dimerization has meaning of the staggered energy density, $\epsilon(x)=e^{i2k_Fx}\vec{S}(x)\cdot\vec{S}(x+a_0)$, where a_0 is the lattice spacing. In the low-energy limit this expression turns into $\epsilon(x)\propto[\vec{J}_R(x)+\vec{J}_L(x)]\cdot\vec{S}_{2k_F}(x')$, where the limit $x'\to x$ must be taken. Short calculation shows that this leads to Eq. (54) above. We now observe that, quite similarly to $\epsilon=h^0$, the helicity operator h^a (with vector index $a=x,y,z$) may be understood as arising from the fusing of the spin current [Eq. (44)] with the $2k_F$ component of the density field: $h^a_{2k_F}(x)=[J^a_R(x)+J^a_L(x)]\rho_{2k_F}(x')$. Here again $x'\to x$ limit is understood. One can check that all relations involving h^a derived above follow from this observation. In particular, we note that

$$h^x_{2k_F} = \frac{2}{\pi a_0}\sin(\sqrt{2\pi}\varphi_\rho + 2k_F x)\sin(\sqrt{2\pi}\theta_\sigma), \tag{63}$$

implying that correlation function of this field decays with the same exponent (K_ρ) as that of $S^x_{2k_F}$ discussed in the beginning of this section. It is useful to note that helicity disappears in the spin-chain limit of the problem, together with the low-energy density fluctuations.

IV. PERTURBATIVE APPROACH

The aim of this section is to show that the limit $\gamma_F \to 0$ can be obtained in a straightforward perturbation expansion in α_R. While results of this section parallel conclusions of the previous *two-subband* consideration in Secs. III and III A, the technical steps involved are somewhat involved and are, in our opinion, of interest in its own right. In addition, similar perturbative consideration of the *impurity* effects later in this work turns out to be very informative for understanding the physics. For these reasons we choose to present the main steps of the perturbation theory in spin-orbit coupling α_R. The calculation starts very similar to Ref. 14 but concludes with quite different steps.

The idea is to treat both magnetic field and Rashba terms as perturbations to the standard single-channel ballistic quantum wire charge and spin sectors of which are described by the decoupled Tomonaga-Luttinger Hamiltonians (35) and (36). The parameters $K_{\rho/\sigma}, u_{\rho/\sigma}$ of these unperturbed harmonic sectors are given by Eq. (41) but *with* $\gamma_F=0$. Spin backscattering term [Eq. (38)] is in principle present (again with $\gamma_F=0$) but will not be required in the subsequent calculation.

Thus the perturbing terms are, see Eq. (3), the Zeeman term,

$$\hat{H}_Z = -\Delta_z \int dx \Psi^\dagger_s \frac{\sigma^z_{ss'}}{2}\Psi_{s'} \tag{64}$$

and the spin-orbit term given by,

$$\hat{H}_R = \alpha_R \int dx \Psi^\dagger_s(x)\sigma^x_{ss'}\left(-i\frac{\partial}{\partial x}\right)\Psi_{s'}(x), \tag{65}$$

where $\Psi_{s=\uparrow,\downarrow}(x)$ right and left movers of unperturbed single-channel quantum wire

$$\Psi_s = R_s e^{ik_F x} + L_s e^{-ik_F x}. \tag{66}$$

Bosonized expressions for R/L operators parallels that in Eqs. (22)–(24) where the subband index $\nu=\pm$ should be replaced by *spin index* $s=\uparrow,\downarrow$. In terms of charge and spin modes introduced in Eq. (33), dual pair φ_s, θ_s for a fermion of a given spin projection s is expressed as

$$\varphi_s = (\varphi_\rho + s\varphi_\sigma)/\sqrt{2}, \quad \theta_s = (\theta_\rho + s\theta_\sigma)/\sqrt{2}, \tag{67}$$

where the following correspondence for the right-hand side of the equations is understood: $s=\uparrow=+1$ and $s=\downarrow=-1$. It then follows that

$$[\varphi_\lambda(x), \theta_{\lambda'}(x')] = \frac{i}{2}[1 - \text{sign}(x-x')], \tag{68}$$

where $\lambda = s = \uparrow,\downarrow$ or ρ, σ.

We then find

$$\hat{H}_Z = -\Delta_z \int dx (J^z_R + J^z_L) = -\frac{\Delta_z}{\sqrt{2\pi}}\int dx \partial_x \varphi_\sigma \tag{69}$$

and

$$\hat{H}_R = 2\alpha_R k_F \int dx [J^x_R(x) - J^x_L(x)]$$

$$= \frac{2\alpha_R k_F \eta_\uparrow \eta_\downarrow}{\pi a_0}\int dx \cos(\sqrt{2\pi}\varphi_\sigma)\sin(\sqrt{2\pi}\theta_\sigma), \tag{70}$$

where J^a_R and J^a_L are the ath components of chiral (right and left) spin currents ($a=x,y,z$) defined as

$$J^a_R = R^\dagger_s(x)\frac{\sigma^a_{ss'}}{2}R_{s'}(x), \quad J^a_L = L^\dagger_s(x)\frac{\sigma^a_{ss'}}{2}L_{s'}(x). \tag{71}$$

Note that $\hat{H}_R$, being determined by the *difference* of right and left spin currents, is odd under spatial inversion $\mathcal{P}$ which interchanges right and left movers.

Finally, we rescale fields $\theta_\sigma \to \theta_\sigma/\sqrt{K_\sigma}$ and $\varphi_\sigma \to \sqrt{K_\sigma}\varphi_\sigma$ and account for the Zeeman term [Eq. (69)] by a position-dependent *shift* $\varphi_\sigma \to \varphi_\sigma + \sqrt{K_\sigma/(2\pi)}\Delta_z x/u_\sigma$ so that

$$\hat{H}_R = \tilde{g}_R \int dx \cos(\sqrt{2\pi K_\sigma}\varphi_\sigma + q_0 x)\sin\left(\sqrt{\frac{2\pi}{K_\sigma}}\theta_\sigma\right),$$

$$\tilde{g}_R = \frac{2\alpha_R k_F}{\pi a_0}\eta_\uparrow\eta_\downarrow, \quad q_0 = \frac{K_\sigma \Delta_z}{u_\sigma}. \tag{72}$$

Up to a factor of $K_\sigma \approx 1$ which appears here due to the rescaling of the bosonic fields above, q_0 matches with δk_F in Eq. (9) for $\alpha_R=0$.

These standard transformations leave Eq. (72) as the only perturbation, and correspond to $\Delta_z \gg \alpha_R k_F$ limit of the theory. Observe that charge sector of the theory, Eqs. (35) and (41) with $\gamma_F=0$, decouples from the spin sector and does not generate any new term.

The calculation proceeds by expanding partition function in powers of g_R

$$Z = \int e^{-S_0}\left(1 - \int d\tau \hat{H}_R + \frac{1}{2}\int d\tau d\tau' \hat{H}^2_R + \dots\right), \tag{73}$$

where the unperturbed action S_0 is particularly simple

$$S_0 = \int dx d\tau \left\{\frac{u_\sigma}{2}[(\partial_x\varphi_\sigma)^2 + (\partial\theta_\sigma)^2] - i\partial_\tau\varphi_\sigma\partial_x\theta_\sigma\right\}. \tag{74}$$

The backscattering term $\cos(\sqrt{8\pi K_\sigma}\varphi_\sigma + 2\delta k_F x)$, see Eq. (38), is ignored as it generates terms at higher order in perturbation theory (by coupling with $\hat{H}^2_R$).

Further details of the calculation are summarized in Appendix B. The result is Eq. (B12) from where we can read off the leading correction to the spin Hamiltonian

$$H_\sigma^{(2)} = \left(\frac{\alpha_R k_F}{K_\sigma \Delta_z}\right)^2 \frac{U(2k_F)}{(\pi a_0)^2 K_\sigma} \int dx \cos\left(\sqrt{\frac{8\pi}{K_\sigma}}\theta_\sigma\right). \quad (75)$$

This compares well with our previous result [Eq. (37)] (remember that we rescaled θ as $\theta \to \theta/\sqrt{K}$ above) in the limit $\gamma_F \ll 1$, where the described calculation is applicable.

It is worth pointing out that perturbatively generated Cooper term, instead of being proportional to $(\alpha_R k_F/v_F)^2$ as one would naïvely expect from a straight forward perturbation expansion of Eq. (72), acquires a nontrivial dependence on both the backscattering amplitude, $U(2k_F)/(2\pi v_F)$, [via $K - 1/K$ combination in Eq. (B11)] and the Zeeman term Δ_z [via q_0 dependence of Eq. (B11)] and is proportional to $(\alpha_R k_F/\Delta_z)^2 U(2k_F)/v_F$. A second thing to notice is the crucial role of Klein terms [see Eq. (B12)] in generating the correct (positive) sign in Eq. (75).

A. Arbitrary angle between SO and magnetic field

Perturbative approach is convenient for analyzing angular stability of the Cooper phase. Let us consider the situation when the angle between the magnetic-field and spin-orbit directions is $\pi/2-\beta$, so that $\beta=0$ corresponds to the orthogonal orientation studied so far in this paper. It is convenient to keep $\hat{z}$ as the direction of magnetic field, in which case the spin-orbital term changes from $\alpha_R \hat{p} \sigma^x$ to $\alpha_R \hat{p}[\sigma^x \cos(\beta) + \sigma^z \sin(\beta)]$. Following steps that led to Eqs. (69), (70), and (72) we obtain that $H_{R,\beta} = H_R^{(x)} + H_R^{(z)}$

$$H_R^{(x)} = \tilde{g}_R^{(x)} \int dx \cos(\sqrt{2\pi K_\sigma}\varphi_\sigma + q_0 x) \sin\left(\sqrt{\frac{2\pi}{K_\sigma}}\theta_\sigma\right),$$

$$H_R^{(z)} = -\sqrt{\frac{2}{\pi}} \alpha_R k_F \sin(\beta) \int dx \partial_x \theta_\sigma, \quad (76)$$

where first (second) equation represents contribution from σ^x (σ^z) matrix. The coupling constant of cosine term is only slightly modified in comparison with Eq. (72), $\tilde{g}_R^{(x)} = \tilde{g}_R \cos^2(\beta)$. $H_R^{(z)}$ represents the main new feature of the nonorthogonal situation. It is naturally absorbed into $H_R^{(x)}$ by a simple *position-dependent* shift of field θ_σ which results in

$$H_{R,\beta} = \tilde{g}_R^{(x)} \int dx \cos(\sqrt{2\pi K_\sigma}\varphi_\sigma + q_0 x) \times \sin\left(\sqrt{\frac{2\pi}{K_\sigma}}\theta_\sigma + \delta x\right);$$

$$\delta = \frac{2\alpha_R k_F \sin(\beta)}{\sqrt{K_\sigma} v_F}. \quad (77)$$

Note that both φ and θ fields acquire oscillations now. The consequence of this is that the generated Cooper term [compare with Eq. (75)]

$$H_\beta^C = \left[\frac{\alpha_R k_F \cos(\beta)}{K_\sigma \Delta_z}\right]^2 \frac{U(2k_F)}{(\pi a_0)^2 K_\sigma} \times \int dx \cos\left(\sqrt{\frac{8\pi}{K_\sigma}}\theta_\sigma + 2\delta x\right), \quad (78)$$

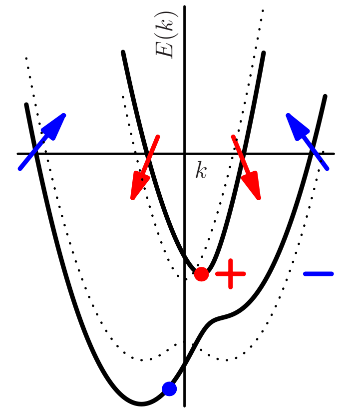

FIG. 4. (Color online) Occupied subbands $\epsilon_\pm$ for the case of nonorthogonal spin-orbital and magnetic-field axes. Arrows illustrate spin polarization, as in Fig. 1. Filled dots indicate location of the center-of-mass for (+) (red) and (−) (blue) subbands. Dashed lines show (±) subbands of Fig. 1, corresponding to the orthogonal orientation, $\beta=0$, for comparison.

does not conserve momentum for $\beta \neq 0$. This important result is evident in the single-particle spectrum, which now reads ($\beta \neq 0$)

$$\epsilon_\pm = \frac{k^2}{2m} \pm \sqrt{(\alpha_R k)^2 + (\Delta_z/2)^2 - \alpha k \Delta_z \sin\beta}, \quad (79)$$

and is to be compared with Eq. (4), which describes $\beta=0$ situation. The spectrum [Eq. (79)] acquires an asymmetry about the energy axis as seen in Fig. 4. The top (+) subband shifts toward the right and the bottom (−) one toward the left. As a result, the pair tunneling does not conserve momentum for $\beta \neq 0$, as is seen in Eq. (78) above.

The momentum mismatch δ destabilizes the Cooper order and eventually, for some critical β_c, destroys it completely. The critical angle can be easily estimated by comparing two spatial scales: ξ_C, which describes *perfect* SDW order, and $1/\delta$, which represents the scale on which momentum nonconservation becomes pronounced. Estimating $\xi_C \sim e^{\ell_c}$, where $\tilde{g}_R(\ell_c) \sim 1$, we obtain that the Cooper order is destroyed when

$$U(2k_F)[\alpha_R k_F \cos(\beta)/\Delta_z]^2 \approx \alpha_R k_F \sin(\beta). \quad (80)$$

Observe that this happens already for small angles, we can simplify the expression for the critical angle

$$\beta_c \propto \frac{U(2k_F)}{\Delta_z} \frac{\alpha_R k_F}{\Delta_z} \ll 1. \quad (81)$$

This angular sensitivity of the found Cooper order to mutual orientation of the magnetic and spin-orbital directions can be used as an experimental probe to differentiate between it and other, spin-orbit independent, many-body instabilities of interacting quantum wire.

V. ISOLATED IMPURITY

SDW_x state manifests itself not only in spin correlations [Eq. (52)]. It turns out that its response to a weak potential scattering (impurity) is rather nontrivial. We consider here a weak delta-function impurity $V(x)=V_0\delta(x)$, with strength V_0, located at the origin. The condition $V_0 \ll \Delta_c$ means that impurity can be considered as a weak perturbation to the established SDW phase. Since the latter is most robust in the limit of $\gamma_F \ll 1$, this is the limit we consider in this section. (The limit of strong impurity, $V_0 \gg \Delta_c$, is rather standard: impurity destroys the SDW and the wire flows into an insulator at low energies.[34])

The interaction of electrons with an impurity potential $V(x)$ is given by

$$\hat{V} = \int dx V(x) \sum_{s=\uparrow,\downarrow} \Psi_s^\dagger(x)\Psi_s(x). \tag{82}$$

As usual, it is the *backscattering* (ρ_{2k_F}) part of the density that has to be considered, $\rho_{2k_F}=\Sigma_s(R_s^\dagger L_s+\text{h.c.})$. Working in the two-subband basis of eigenstates $\nu=\pm$ of the noninteracting Hamiltonian which includes Rashba spin-orbit term and the Zeeman term, see Eq. (14), we derive for the $2k_F$ component of the density $\rho(x)$ at the origin

$$\rho_{2k_F} = \rho_{2k_F}^{\text{intra}} + \rho_{2k_F}^{\text{inter}}, \tag{83}$$

where the *intra*subband part is

$$\rho_{2k_F}^{\text{intra}} = \cos(\gamma_F)(L_+^\dagger R_+ + L_-^\dagger R_- + \text{h.c.}), \tag{84}$$

while the novel *inter*subband contribution is present due to the nonorhogonality of spin states in $(\pm)$ subbands discussed in Sec. II,

$$\rho_{2k_F}^{\text{inter}} = \sin[\gamma_F](L_-^\dagger R_+ - L_+^\dagger R_- + \text{h.c.}). \tag{85}$$

Alternatively, one can think of these two contributions as representing $\tilde{\rho}$ and $\tilde{h}^y$ contributions in Eq. (60). In terms of bosonic fields the density reads

$$\rho_{2k_F}(x=0) = \frac{-2\sin(\sqrt{2\pi}\varphi_\rho)}{\pi a_0}[\cos[\gamma_F]\cos(\sqrt{2\pi}\varphi_\sigma) - \sin(\gamma_F)\cos(\sqrt{2\pi}\theta_\sigma)]. \tag{86}$$

We now observe that setting $\theta_\sigma \to \theta_\sigma^{\text{cl}}=(m+\frac{1}{2})\sqrt{\pi/2}$, as appropriate for the Cooper phase (Sec. III C), *nullifies* the backscattering component of the density [Eq. (86)]. The first (intrasubband) term gets killed by diverging fluctuations of φ_σ, *dual* to θ_σ. Intriguingly, the second (intersubband) contribution is also zero because $\cos[\sqrt{2\pi}\theta_\sigma^{\text{cl}}]=\pm\cos[\pi/2]=0$.

This argument can be made more precise by following calculations described in Refs. 29, 35, and 36. We parametrize $\theta_\sigma=\theta_\sigma^{\text{cl}}+\delta\theta$ and expand the relevant cosine term in Hamiltonian (37) to second order in fluctuations $\delta\theta$. One obtains a *massive* term $\propto\Delta_c(\delta\theta)^2$ in the Hamiltonian, which causes exponential decay in correlation functions of the dual φ_σ field. In particular $\langle\cos\sqrt{2\pi}\varphi_\sigma(0,\tau)\cos\sqrt{2\pi}\varphi_\sigma(0,\tau')\rangle$ will decay as $\exp[-\Delta_c|\tau-\tau'|]$. Thus for an incoming particle with energy $\omega \ll \Delta_c$, the intersubband scattering channel is absent.

Substituting $\theta_\sigma=\theta_\sigma^{\text{cl}}+\delta\theta$ in the second term in Eq. (86) converts it into $\gamma_F\sin(\sqrt{2\pi}\varphi_\rho)\sin(\sqrt{2\pi}\delta\theta)$. Correlations of $\sin[\sqrt{2\pi}\delta\theta]$ are also short-ranged

$$\langle\sin[\sqrt{2\pi}\delta\theta(\tau)]\sin[\sqrt{2\pi}\delta\theta(\tau')]\rangle \approx \frac{\Delta_c}{v_F k_F}\times\sinh[K_0(\Delta_c|\tau-\tau'|)], \tag{87}$$

where $K_0(x)\sim e^{-x}/\sqrt{x}$ (for $x\gg 1$) is the modified Bessel function (see Ref. 29). The two exponentially decaying contributions add up (in second-order perturbation theory in impurity strength V_0) to produce an effective two-particle backscattering potential $\propto(V_0^2/\Delta_c)\cos[\sqrt{8\pi}\varphi_\rho(0,\tau)]$. This generated *two-particle* impurity backscattering term, however, is relevant only for strongly repulsive interactions, $K_\rho<1/2$. We are thus left with *irrelevant* impurity potential for as long as $V_0 \ll \Delta_c$ is justified and for not too strong repulsion, $1/2 \le K_\rho \le 1$: SDW_x state is not sensitive to weak disorder!

The situation is similar to that in recently proposed edge states in quantum spin Hall system.[37] There, gapless spin-up and spin-down excitations propagate in opposite directions along the edge, which forbids single-particle backscattering. Interacting electrons, however, can backscatter off the impurity in pairs.[38,39]

Our conclusion $\rho_{2k_F}\to 0$, Eq. (86), rests on somewhat technical condition $\cos(\sqrt{2\pi}\theta_\sigma^{\text{cl}})=0$ and deserves a better understanding. This is provided by the *perturbative* calculation in Appendix D. The idea of the calculation is similar to that in Sec. IV: treat both impurity and *spin-orbit* terms as perturbation and generate spin-orbit-related corrections to backscattering potential [proportional to γ_F in Eq. (86)] *perturbatively*. In this way we can be certain that *all* symmetry-allowed contributions are accounted for. This argument also makes it clear that the generated terms, being produced by the SOI, have to be *odd* under spatial inversion $\mathcal{P}$. It is this symmetry that guaranties that $\sin[\sqrt{2\pi}\theta_\sigma]$ cannot be generated in the process. This instructive calculation, carried in Appendix D, indeed supports the conclusions of the more formal *two-subband* approach described in this section.

We conclude this section by pointing out experimental consequences of our findings. Suppression of single-particle backscattering off weak impurity under the outlined conditions implies an unusual *negative* magnetoresistance in one-dimensional wire. Indeed, the conductance of the wire with such an impurity should remain at the perfect $G_{\text{wire}}=2e^2/h$ value for as long as applied magnetic field is directed perpendicular to the spin-orbital (σ^x here) axis. By either turning the magnetic field off, or simply rotating the sample (so that momentum mismatch between the two subbands destabilizes SDW_x order), one should observe that conductance plateau deteriorates. It will be destroyed completely in the zero-temperature limit.

VI. HEISENBERG ANTIFERROMAGNET WITH DM INTERACTION AND MAGNETIC FIELD

In this section we will consider a closely related problem: the effect of uniform and staggered Dzyaloshisnkii-Moriya interaction on a spin $S=1/2$ Heisenberg antiferromagnet (HAFM) in the presence of a magnetic field perpendicular to the DM vector. We start by demonstrating the equivalence between the quantum wire problem, analyzed in Secs. II–V, with that of Heisenberg spin chain subject to an additional uniform DM interaction. We then present a novel *chiral rotation* argument to show that results of the two-subband approach in Sec. III can be obtained from a straightforward combination of two independent rotations for right and left movers. We also discuss the difference between the *uniform* and the *staggered* DM interaction, analyzed previously in Ref. 27.

A. Weak uniform DM interaction and strong magnetic field

The Hamiltonian of an isotropic HAFM spin chain is given by

$$H_{\rm heis} = J\sum_j (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + S_j^z S_{j+1}^z). \tag{88}$$

In the continuum limit, the above Hamiltonian is most conveniently described in terms of the $SU(2)_1$ Wess-Zumino-Novikov-Witten model, the basic ingredients of which are the uniform $SU(2)$ left $\vec{J}_L(x)$ and right $\vec{J}_R(x)$ spin currents, defined via fermions in Eq. (71), and the staggered magnetization $\vec{N}(x)$.[30] Another important component is provided by *staggered* energy density (dimerization), see Eq. (54), but this does not develop any order in the presence of external magnetic field. These operators will be used to represent the continuum limit of the spin,

$$\vec{S}_j \to a_0[\vec{J}_L(x) + \vec{J}_R(x) + (-1)^{x/a_0}\vec{N}(x)], \tag{89}$$

where a_0 is the lattice spacing and continuous space variable is introduced via $x=ja_0$. Continuum limit of Eq. (88) is $H_{\rm heis}=H_0+H_{\rm bs}$, where

$$H_0 = \frac{2\pi v}{3}\int dx(\vec{J}_R\cdot\vec{J}_R + \vec{J}_L\cdot\vec{J}_L) \tag{90}$$

$$= \frac{v}{2}\int dx[(\partial_x\varphi)^2 + (\partial_x\theta)^2]. \tag{91}$$

The first line constitutes non-Abelian spin current formulation of the problem, which is convenient for analyzing spin-rotation invariant [SU(2)] problems, while the second makes connection with familiar Abelian bosonization result [Eq. (36)]: identifications $(\varphi_\sigma,\theta_\sigma)\to(\varphi,\theta)$, $u_\sigma \to v = \pi J a_0/2$, and $k_F \to \pi/(2a_0)$ finalize the connection. The marginal backscattering term accounts for residual interaction between spin excitations,

$$H_{\rm bs} = -g_{\rm bs}\int dx \vec{J}_R\cdot\vec{J}_L. \tag{92}$$

Its coupling constant $g_{\rm bs}>0$ is known from extensive numerical studies of the Heisenberg chain:[40] $g_{\rm bs}=0.23\times 2\pi v$. This term is responsible for the fact that initial value of the Luttinger parameter $K_\sigma=1+g_{\rm bs}/2\pi v$ is greater than 1.

There are no low-energy charge fluctuations as the charge is "locked" to the lattice by relevant Umklapp processes. This allows one to replace cosine of charged boson φ_ρ in Eq. (52) by its expectation value λ: $\lambda=\langle\cos[\sqrt{2\pi}\varphi_\rho]\rangle$. The first line of Eq. (52) then establishes Abelian representation of the staggered magnetization

$$\begin{pmatrix} N^x \\ N^y \\ N^z \end{pmatrix} = \frac{\lambda(-1)^x}{\pi a_0}\begin{bmatrix} -\sin(\sqrt{2\pi}\theta) \\ \cos(\sqrt{2\pi}\theta) \\ -\sin(\sqrt{2\pi}\varphi) \end{bmatrix}. \tag{93}$$

Magnetic field is introduced via Zeeman term

$$H_Z^{(z)} = -\sum_j g\mu_B B S_j^z \to -\Delta_z\int dx(J_R^z + J_L^z), \tag{94}$$

which is identical to Eq. (69). The small upper index (z here) indicates the axis in spin space. Finally, the Dzyaloshinskii-Moriya Hamiltonian describes asymmetric and odd under spatial inversion interaction between spins

$$H_{\rm DM}^{(x)} = \sum_j \vec{D}\cdot\vec{S}_j\times\vec{S}_{j+1}. \tag{95}$$

Vector $\vec{D}=D\hat{x}$ fixes direction of spin anisotropy, which we choose to be along spin-$\hat{x}$ direction, similar to the spin-orbit direction choice in Eq. (3).

Using Eq. (89) we see that continuum limit of the uniform DM Hamiltonian requires the knowledge of the following operators

$$O_J(x,x') = J^y(x)J^z(x') - J^z(x)J^y(x'),$$

$$O_N(x,x') = N^y(x)N^z(x') - N^z(x)N^y(x'), \tag{96}$$

where $x'=x+a_0$. The first of these follows from the well-known OPE for spin currents, see for example Eqs. (25) and (26) of Ref. 41 and set $\bar{z}-\bar{z}'=ia_0$:

$$O_J(x,x') = \frac{1}{\pi a_0}[J_R^x(x) - J_L^x(x)]. \tag{97}$$

The second requires more work and is calculated using the definition (93) and bosonic OPEs (B1)

$$O_N(x,x') = \frac{2\lambda^2}{\pi a_0}[J_R^x(x) - J_L^x(x)]. \tag{98}$$

As a result

$$H_{\rm DM}^{(x)} = \tilde{D}\int dx(J_R^x - J_L^x), \quad \tilde{D} = \frac{Da_0}{\pi}(1+2\lambda^2). \tag{99}$$

Comparing Eq. (99) with Eqs. (70) and (94) with Eq. (69) we observe that the problem at hand is identical to that analyzed in Sec. IV. We immediately conclude that the $2k_F=\pi$ (staggered) components of spins acquire long-range order along the direction of $\vec{D}$ as soon as magnetic field is turned on along the orthogonal direction.

$$\langle N^x\rangle = \text{const}(-1)^x, \quad \langle N^{y,z}\rangle = 0. \tag{100}$$

It is useful to compare this equation with Eq. (52). Note that the effect is controlled by the backscattering amplitude $g_{\rm bs}=2(K_\sigma-1)$.

Except for the brief remark in Sec. VIII of Ref. 42, the possibility of the long-range order in the magnetized spin chain with asymmetric uniform DM interaction has not been discussed in the literature, to the best of our knowledge. We also would like to note technical similarities of our problem with that of an anisotropic Heisenberg chain in a transverse field, considered in Ref. 43.

Experimentally, the field-induced long-ranged magnetic order would be probably easiest to observe via thermodynamic measurements. Similar to the case of copper benzoate,[26] the order should show up via an exponential suppression of the specific heat. The magnitude of this suppression, which is controlled by the energy gap Δ_c [Eq. (51)], should be a sensitive function of the magnetic field orientation, reaching maximum when the field is orthogonal to the DM axis as discussed above.

B. Chiral rotation and tilted magnetic field

We now present an elegant argument, borrowed from the technically closely related study of quantum kagomé antiferromagnet,[44] exposing the nature of the Cooper order to the fullest. Consider the situation, discussed in Sec. IV A, of the *tilted* magnetic field making an angle $\pi/2-\beta$ with the DM axis direction (which we keep at $\hat{x}$). The Hamiltonian describing this arrangement is given by $H=H_0+H_{\rm bs}+V$, where V includes Zeeman and DM fields

$$V=\int dx[d_R J_R^x - d_L J_L^x - h_1(J_R^z+J_L^z)]. \tag{101}$$

The parameters are

$$d_R=\tilde{D}-h_2, \quad d_L=\tilde{D}+h_2, \quad h_1=h\cos\beta, \;\; h_2=h\sin\beta. \tag{102}$$

We now rotate right (left) spin currents $\vec{J}_R$ ($\vec{J}_L$) about $\hat{y}$ axis by angle θ_R (θ_L) such that after the rotation the V term becomes

$$V=-\int dx(\sqrt{d_R^2+h_1^2}M_R^z+\sqrt{d_L^2+h_1^2}M_L^z). \tag{103}$$

The relation between old ($\vec{J}$) and new ($\vec{M}$) currents is simple

$$\vec{J}_R=\mathcal{R}(\theta_R)\vec{M}_R, \quad \vec{J}_L=\mathcal{R}(\theta_L)\vec{M}_L, \tag{104}$$

here $\mathcal{R}$ is the rotation matrix

$$\mathcal{R}(\theta)=\begin{bmatrix}\cos(\theta) & 0 & -\sin(\theta)\\ 0 & 1 & 0\\ \sin(\theta) & 0 & \cos(\theta)\end{bmatrix}. \tag{105}$$

The rotation angles are given by $\tan(\theta_R)=d_R/h_1$ and $\tan(\theta_L)=-d_L/h_1$. The key reason behind these rotations is the observation that unperturbed Hamiltonian H_0, being the sum of commuting right and left terms, is invariant under the rotations. However, the backscattering $H_{\rm bs}$ is not, and transforms into

$$H_{\rm bs}=-g_{\rm bs}\int dx[\cos(\gamma)(M_R^xM_L^x+M_R^zM_L^z) + \sin(\gamma)(M_R^xM_L^z-M_R^zM_L^x)+M_R^yM_L^y], \tag{106}$$

where $\gamma=\theta_R-\theta_L$. The reason for this transformation is of course that right and left currents are rotated in opposite directions (and by different amounts). Observe that Eq. (106) matches the interaction part of Eq. (47).

The nice thing about the rotation is that now both SO and Zeeman fields can be taken into account by simple linear transformations of spin bosons. Indeed, using Abelian bosonization we observe that

$$V=-\int dx\left(\frac{vt_\varphi}{\sqrt{2\pi}}\partial_x\varphi_\sigma+\frac{vt_\theta}{\sqrt{2\pi}}\partial_x\theta_\sigma\right), \tag{107}$$

where

$$t_\varphi=(\sqrt{d_L^2+h_1^2}+\sqrt{d_R^2+h_1^2})/2v,$$
$$t_\theta=(\sqrt{d_L^2+h_1^2}-\sqrt{d_R^2+h_1^2})/2v. \tag{108}$$

These linear terms are removed by shifts

$$\varphi_\sigma\to\varphi_\sigma+\frac{t_\varphi x}{\sqrt{2\pi}},$$

$$\theta_\sigma\to\theta_\sigma+\frac{t_\theta x}{\sqrt{2\pi}}. \tag{109}$$

The price is that transverse components $M_{R/L}^{x,y}$ acquire oscillating position-dependent factors

$$M_R^+\to M_R^+e^{-i(t_\varphi-t_\theta)x}, \quad M_L^+\to M_L^+e^{i(t_\varphi+t_\theta)x}. \tag{110}$$

The major consequence of this is that every term in $H_{\rm bs}$, with a single exception of $M_R^zM_L^z$ one, picks up oscillating factor

$$H_{\rm bs}=-g_{\rm bs}\int dx\Bigg\{\cos(\gamma)M_R^zM_L^z+\frac{\cos(\gamma)-1}{4}\times(M_R^+M_L^+e^{i2t_\theta x}+\text{h.c.})+\frac{\cos(\gamma)+1}{4}(M_R^+M_L^-e^{-i2t_\varphi x}+\text{h.c.}) + \frac{\sin(\gamma)}{2}[M_L^zM_R^+e^{-i(t_\varphi-t_\theta)x}-M_R^zM_L^+e^{i(t_\varphi+t_\theta)x}+\text{h.c.}]\Bigg\}. \tag{111}$$

This equation contains *all* arrangements that we have discussed in this paper. Setting $t_\theta=0$ corresponds to *orthogonal* ($\beta=0$) orientation of spin-orbit (DM) and magnetic-field axis. In that limit we recover Eqs. (45) ($\vec{J}$ there corresponds to $\vec{M}$ here). Allowing for $\beta>0$ leads us to Sec. IV A, results of which represent Abelian version of the discussion here. As shown there, the SDW_x phase is stable in a finite angular interval near $\beta=0$.

It is worth noting that Eqs. (103) and (107) imply finite expectation values of $M_{R/L}^z$ currents:

$$\langle M_{R/L}^z\rangle=\sqrt{d_{R/L}^2+h_1^2}/(4\pi v). \tag{112}$$

By virtue of Eq. (105) this implies finite expectation values of the uniform magnetization along z and x axis, $J_R^{z/x}+J_L^{z/x}$, and uniform spin current, $J_R^{z/x}-J_L^{z/x}$, along these two axes. That is,

$$\langle J_R^x-J_L^x\rangle=\sin(\gamma)\frac{t_\phi}{2\pi}, \quad \langle J_R^z-J_L^z\rangle=-\cos(\gamma)\frac{t_\theta}{2\pi},$$

$$\langle J_R^x + J_L^x \rangle = -\sin(\gamma)\frac{t_\theta}{2\pi}, \quad \langle J_R^z + J_L^z \rangle = \cos(\gamma)\frac{t_\phi}{2\pi}. \tag{113}$$

Equilibrium values of spin current and magnetization along the y direction vanish. These relations complement discussion of the relations between staggered ($2k_F$) components of various fields in Sec. III C.

C. Strong uniform DM interaction and weak magnetic field

It is instructive to consider another "tricky" limit of the problem: strong uniform DM interaction and a weak magnetic field, $D \gg \Delta_z$. The idea here is to account for the DM term *exactly* on the lattice level, see Eq. (114), and treat the Zeeman term as a small perturbation to the obtained Hamiltonian (116).

The two axis, DM and Zeeman, are still orthogonal but it is more convenient now to let $\vec{D}$ point along the $\hat{z}$ axis in spin space, while magnetic field is pointing along $\hat{x}$. Following Ref. 42 we perform unitary transformation of the *lattice* Hamiltonian to absorb the uniform DM term exactly,

$$S_j^+ \to \tilde{S}_j^+ e^{i\alpha j}, \quad S_j^- \to \tilde{S}_j^- e^{-i\alpha j} \text{ and } S_j^z \to \tilde{S}_j^z, \tag{114}$$

where $\alpha = \arctan(D/J) \approx D/J$ for $D \ll J$. The transformed Hamiltonian is that of the XXZ spin chain with weak anisotropy

$$\tilde{H} = \tilde{J}\sum_j \left(\tilde{S}_j^x \tilde{S}_{j+1}^x + \tilde{S}_j^y \tilde{S}_{j+1}^y + \frac{J}{\tilde{J}} \tilde{S}_j^z \tilde{S}_{j+1}^z \right), \tag{115}$$

where $\tilde{J} = \sqrt{J^2 + D^2}$. Magnetic field term, however, acquires position dependence under this transformation

$$\tilde{H}_Z^{(x)} = -\frac{\Delta_z}{2}\sum_j (\tilde{S}_j^+ e^{i\alpha j} + \tilde{S}_j^- e^{-i\alpha j}). \tag{116}$$

Bosonizing it according to the rules described in Secs. III A and IV, we obtain

$$\tilde{H}_Z^{(x)} = \frac{\eta_\uparrow \eta_\downarrow \Delta_z}{\pi a_0} \int dx \sin(\sqrt{2\pi K}\varphi)\cos\left(\sqrt{\frac{2\pi}{K}}\theta + \alpha x \right), \tag{117}$$

where Luttinger parameter of the XXZ chain [Eq. (115)] is given by $K^{-1} = 1 - \arccos[J/\sqrt{J^2+D^2}]/\pi \approx 1 - D/(\pi J)$. The shift of θ field by αx can be easily understood from Eq. (99), adapted for the DM axis along $\hat{z}$ direction in spin space: $H_{\rm DM}^{(z)} = \tilde{D}\int dx (J_R^z - J_L^z) \propto \tilde{D}\int dx \partial_x \theta$ which indeed can be accounted for by shifting θ.

Being of nonzero conformal spin, the Zeeman term (117) generates under RG two new terms of importance to us: $\tilde{H}_\varphi \propto (\Delta_z^2 D/J)\cos(\sqrt{8\pi K}\varphi)$, of scaling dimension $2K = 2 + 2D/(\pi J) > 2$, and $\tilde{H}_\theta = G_\theta \int dx \cos(\sqrt{\frac{8\pi}{K}}\theta + 2\alpha x)$ with scaling dimension $2/K = 2 - 2D/(\pi J) < 2$. The position-dependent phase $2\alpha x$, nonetheless, makes it *irrelevant*.

Indeed, the coupling constant can be estimated, following momentum-shell RG of Appendix C, as $G_\theta \propto (\Delta_z/v)^2 (K - 1/K)^2 \approx (\Delta_z D/J)^2$, see Eqs. (C7) and (C12). Neglecting the oscillating phase for the moment, we can easily estimate the correlation length corresponding to the Cooper order as $\xi_c = \exp(\ell_c)$, where $G_\theta(\ell_c) = G_\theta(0)\exp[(2-2/K)\ell_c] \approx 1$, so that $\ell_c = (\pi J/D)\ln[J/(\Delta_z D)]$.

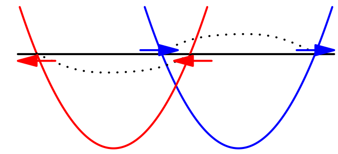

FIG. 5. (Color online) Band structure in the absence of magnetic field. Note the absence of the gap between the subbands at $k=0$.

This length is to be compared with $\xi_\alpha = 1/\alpha \approx J/D$, which is the length on which the cosine in $\tilde{H}_\theta$ changes *sign*. We observe that

$$\ln(\xi_c) = \frac{\pi J}{D}\ln\left(\frac{J}{\Delta_z D}\right) \gg \ln[\xi_\alpha] = \ln\left(\frac{J}{D}\right), \tag{118}$$

which implies that fast oscillations make it impossible for the $\tilde{H}_\theta$ to reach strong coupling. Hence no order can come from this term.

The other term, $\tilde{H}_\varphi$, *appears* to be irrelevant and it is tempting to disregard it altogether. One, however, must be careful and recall discussion in Sec. III B. Particularly, observe that $\tilde{H}_\varphi$ may fall in region II (with $y_\sigma(0) < 0$). Whether or not this happens is in principle a subject of precise calculation which however can be avoided here. Indeed analysis of Sec. VI B, taken together with results of Sec. III B, shows that Cooper instability develops for arbitrary γ as long as $t_\theta = 0$.

We thus conclude that $\tilde{H}_\varphi$ *must* belong to region II in Fig. 3, even though this conclusion is not at all obvious from the Abelian bosonization analysis sketched above. We chose to present this subtle point in order to demonstrate the power of non-Abelian current formulation in Sec. VI B.

Note finally that $\tilde{H}_\varphi$ still describes Cooper order even though it is written in terms of φ field. The reason for this illustrates another shortcoming of Abelian formulation. Unitary transformation [Eq. (114)] is the lattice version of rotation of right (left) currents by $\theta_R = \pi/2$ ($\theta_L = -\pi/2$) in Sec. IV. (In this limit $\Delta_z = 0$ and magnetic field is to be added later as a perturbation.) Such a rotation corresponds to $\gamma = \pi$ which replaces standard backscattering ($M_R^+ M_L^- + \text{h.c.}$) with ($M_R^+ M_L^+ + \text{h.c.}$). In terms of Abelian bosonization this corresponds to interchange $\cos\sqrt{8\pi}\varphi \leftrightarrow \cos\sqrt{8\pi}\theta$. Moreover, this same notation changes the sign of $M_R^z M_L^z$ term, which corresponds to interchange $K \leftrightarrow 1/K$, which completes the duality mapping. Observe again the ease with which current formulation of Sec. VI B leads to conclusion.

In simpler terms, accounting for DM (or, equivalently, spin-orbit) term only amounts to working in the basis of eigenstates of operator σ_x. These eigenstates are described simply by two *horizontally* shifted parabolas in Fig. 5. Denoting the right (left) subbands by indices 1 ($|1\rangle = |\rightarrow\rangle$) and 2 ($|2\rangle = |\leftarrow\rangle$), we immediately observe that Cooper scattering discussed previously corresponds to the process such as $R_1^\dagger L_1 L_2^\dagger R_2$. (So that initial pair $L_1 R_2$ scatters to final $R_1 L_2$ state. Note that the other possibility, $L_1 \to L_2$ and $R_2 \to R_1$ is forbidden by orthogonality of single-particle basis states, $\langle 1|2\rangle = \langle \to | \leftarrow \rangle = 0$.) Bosonizing this in a standard way we find that {recall $R_1^\dagger L_1 \sim \exp[-i\sqrt{4\pi}(\phi_{1R}+\phi_{1L})]$} $R_1^\dagger L_1 L_2^\dagger R_2 \sim \exp[-i\sqrt{4\pi}(\varphi_1-\varphi_2)] = \exp(-i\sqrt{8\pi}\varphi_\sigma)$. Thus indeed the Cooper process is written in terms of φ_σ field in the new basis.

Observe that this conclusion is special to the limit of no magnetic field, $\Delta_z=0$. As soon as the field is on, $\Delta_z \neq 0$, the band structure changes and horizontally shifted subbands (1,2) turn into vertically shifted pair $(-,+)$ used in this paper (Fig. 1): arbitrary small B leads to the gap Δ_z at the point $k=0$, see Eq. (4). We thus conclude once again that $\Delta_z=0$ is a rather singular limit, not much suited for the consideration of general situation with both spin-orbit and Zeeman fields present.

D. HAFM chain with staggered DM interaction and magnetic field

For the sake of completeness and uniformity of the presentation, we revisit here the well-studied case of interplay between *staggered* DM interaction and Zeeman magnetic field, described in Refs. 27 and 45.

Staggered DM interaction along $\hat{x}$ axis is described by

$$H_{\rm sDM}^{(x)} = D\sum_j (-1)^j (S_j^y S_{j+1}^z - S_j^z S_{j+1}^y). \tag{119}$$

Its continuum limit requires the knowledge of

$$O_{JN}(x,x') = N^y(x)J^z(x') + J^z(x)N^y(x') - J^y(x)N^z(x') - N^z(x)J^y(x'), \tag{120}$$

where, as before in Eq. (96), we set $x'=x+a_0$. It is an easy calculation, using Eq. (26) of Ref. 41, to find that in the absence of magnetic field, $O_{JN}=0$ identically. The situation changes when magnetic field (along $\hat{z}$ axis) is present. Using bosonization, magnetic field is accounted for [see Eq. (72) and line above it] by the shift $\sqrt{2\pi K}\varphi \to \sqrt{2\pi K}\varphi + q_0 x$. Moreover, the main effect of the field here is contained in $q_0 = \Delta_z/v$ and we can keep $K=1$ in all calculations below. Observing that a shift in φ implies equal shifts in $\phi_{R/L}$ fields (so that θ does not change) and working backward through Eqs. (52), (71), and (22) we conclude, following Ref. 27, that in the presence of the magnetic field, spin excitations *along* the field ($\hat{z}$ axis here) and *transverse* to it ($\hat{x},\hat{y}$ axis) have minima at different momenta. Namely, while J^z is still centered at $q=0$, *transverse* components of spin current $J^{x,y}$ acquire minima at $q=\pm q_0 = \pm 2\pi m$, where m is the magnetization. In addition, N^z shifts from π to $q=\pi \pm 2\pi m$ while staggered transverse components $N^{x,y}$ remain at $2k_F=\pi$ point. As a result, the product $N^y J^z$ [first line in (120)] retains its zero-field structure and continues to remain at zero. At the same time the other combination, $J^y N^z$ [second line in (120)], splits into *slow* ($e^{iq_0(x-x')}$) and *fast* ($e^{iq_0(x+x')}$) oscillating pieces which do not cancel each other anymore. The remaining calculation is most conveniently performed using fermionic representations of spin currents [Eq. (71)] and longitudinal magnetization $N^z = \frac{1}{2}[R_\uparrow^\dagger L_\uparrow - L_\downarrow^\dagger R_\downarrow]e^{-iq_0 x} + \text{h.c.}$. Fusing right (left) movers of like spin using Eq. (D4) we obtain, for example,

$$[J_R^+(x) + J_L^+(x)]N^z(x') = -\frac{\sin[q_0(x-x')]}{2\pi(x-x')}(R_\uparrow^\dagger L_\downarrow + L_\uparrow^\dagger R_\downarrow). \tag{121}$$

Bosonizing this expression [note that it is not sensitive to the *sign* of the coordinate difference $(x-x')$] we finally obtain

$$O_{JN}(x,x') = -\frac{q_0\lambda}{2\pi^2 a_0}\cos[\sqrt{2\pi}\theta(x)]. \tag{122}$$

The same result can be obtained, after somewhat longer calculation, using bosonized forms of spin currents and magnetization from the very beginning. The continuum limit of staggered DM term then follows

$$H_{\rm sDM}^{(x)} = \frac{Dq_0\lambda}{\pi^2 a_0}\int dx\, \cos[\sqrt{2\pi}\theta(x)]. \tag{123}$$

This is a highly relevant operator (scaling dimension 1/2), the coupling constant $G_{s-\rm dm}$ of which grows as $G_{s-\rm dm}(\ell) = G_{s-\rm dm}(0)\exp(3\ell/2)$. This, and the dependence of its initial value on the combination $D\Delta_z$ (which enters via dependence on q_0), leads to the energy gap in the system that scales as $(D\Delta_z)^{2/3}$, exactly as Ref. 27 found originally. Note finally that Eqs. (123) and (52) imply that the spins order (in a staggered way) along $\hat{y}$ axis, orthogonal to both DM and magnetic-field directions.

Note that although we have treated staggered DM term [Eq. (119)] as a perturbation to the spin chain subject to magnetic field (which comes in via the momentum q_0 here), we obtained the same strongly relevant result [Eq. (123)] as the authors of Ref. 27 did. However, Ref. 27 arrived at it from a different limit: the authors used *staggered* version of Eq. (114) to account for DM piece [Eq. (119)] exactly and then added magnetic field as a perturbation. The outcome [Eq. (123)] is obtained in both cases simply because it is more relevant (dimension 1/2) than the magnetic field (Zeeman) term (dimension 1): no matter how small $G_{s-\rm dm}$ initially is, it controls the physics. The N^y-order is present for any ratio of $G_{s-\rm dm}/\Delta_z$ ratio.

VII. IMPLICATIONS FOR ESR EXPERIMENTS

In recent years, electron-spin-resonance (ESR) technique has become a leading experimental candidate for probing anisotropic terms in the spin chain. Recent theoretical impetus to this field has been provided in an important work by Oshikawa and Affleck[46,47] who discussed limitations of earlier theoretical work[48–50] and improved and extended upon them by using powerful modern theoretical techniques. The emphasis in Ref. 47 has been to study the role of anisotropic terms, in particular the staggered DM interaction, see Sec. VI D, and exchange anisotropy terms, in modifying the resonance position and the line width. However, ESR in a spin chain with uniform DM interaction was not considered. We will follow a closely related work by De Martino *et al.*,[51,52] who considered ESR for carbon nanotubes with spin-orbit coupling, to investigate the modifications brought upon by the uniform DM term on the ESR spectra.

We consider Faraday configuration in which the static magnetic field and oscillating field are orthogonal to each other. The oscillating driving field (of frequency ω) is in the microwave frequency regime and for all practical purposes the spatial modulation of the field can be ignored. The ESR intensity is given by the transverse (to the static magnetic field) spin structure factor at $q=0$ and frequency ω,

$$I(\omega) = \int dtdx e^{i\omega t} \sum_{r,r'=R/L} \langle J_r^+(x,t) J_{r'}^-(0,0)\rangle. \qquad (124)$$

As before, the Hamiltonian, $H=H_0+H_{\rm bs}+V$, consists of the free part [Eq. (90)], the back scattering part [Eq. (92)], and the DM and Zeeman terms contained in the V term [Eq. (101)]. The angle between DM and magnetic-field directions is $\pi/2-\beta$, see Sec. VI B.

Our aim here is to present the basic picture of ESR response for the case of the spin chain with *uniform* DM interaction (equivalently, quantum wire with spin-orbit interaction). For this *zeroth order* description we omit the backscattering interaction [Eq. (92)] between the spin currents altogether. Under this drastic approximation right $\vec{J}_R$ and left $\vec{J}_L$ spin currents are decoupled. This implies the intensity [Eq. (124)] is the sum of right and left contributions, $I(\omega)=I_R(\omega)+I_L(\omega)$.

To account for the simultaneous presence of DM and Zeeman terms, we now rotate the right and left currents about the y axis as described in Sec. VI B, see in particular Eqs. (104) and (105). As the backscattering [Eq. (106)] is neglected, the full Hamiltonian is given by

$$H = \int dx \left[\sum_{a=x,y,z} \frac{2\pi v}{3} (M_R^a M_R^a + M_L^a M_L^a) - \lambda_R M_R^z - \lambda_L M_L^z \right], \qquad (125)$$

where $\lambda_{R/L}=\sqrt{d_{R/L}^2+h_1^2}=v(t_\varphi \mp t_\theta)$.

We now focus on the contribution of the right spin currents,

$$I_R(\omega) = \int dtdx e^{i\omega t} \langle J_R^+(x,t) J_R^-(0,0)\rangle, \qquad (126)$$

which, in terms of the rotated currents, reads

$$I_R(\omega) = \int dtdx e^{i\omega t} [\cos^2 \theta_R \langle M_R^x(x,t) M_R^x(0)\rangle + \langle M_R^y(x,t) M_R^y(0)\rangle + \sin^2 \theta_R \langle M_R^z(x,t) M_R^z(0)\rangle]. \qquad (127)$$

The cross terms of the kind $\langle M_R^x(x,t) M_R^z(0)\rangle=0$ due to the absence of coupling between x and z components in Hamiltonian (125). Further, switching to $M_R^\pm$ combinations of the currents, the intensity becomes

$$I_R(\omega) = \int dtdx e^{i\omega t} \left[\sin^2 \theta_R \langle M_R^z(x,t) M_R^z(0)\rangle + \frac{(\cos^2 \theta_R - 1)}{4} \langle M_R^+(x,t) M_R^+(0) + \text{h.c.}\rangle + \frac{(\cos^2 \theta_R + 1)}{4} \langle M_R^+(x,t) M_R^-(0) + \text{h.c.}\rangle \right]. \qquad (128)$$

The second line in this expression contributes zero as it requires anomalous averages of the kind $\langle \Psi_{R,\uparrow}(x,t) \Psi_{R,\uparrow}(0,0)\rangle$ which are absent in Eq. (125). We now absorb "right" magnetic field λ_R via the shift of right boson so that $M_R^+ \to M_R^+ e^{-i\lambda_R x/v}$ [compare with Eq. (110)]. It is worth noting that by rotating the spin currents we have mapped the problem of the spin chain with uniform DM and magnetic fields to that of the chain in field λ_R (λ_L) for its right (left) moving components. This allows to borrow results of Ref. 47 and conclude that the first line in Eq. (126) does not contribute to I_R while the last line gives

$$I_R(\omega) = \frac{(\cos^2 \theta_R + 1)}{2} \omega \delta(\omega - |\lambda_R|). \qquad (129)$$

Clearly the contribution of the left-moving sector is obtained by replacing $R \to L$ in the expression above. Thus the ESR signal consists of *two* sharp lines, as previously discussed for the case of carbon nanotube in Refs. 51 and 52,

$$I(\omega) = (\cos^2 \theta_R + 1)\lambda_R \delta(\omega - |\lambda_R|) + (\cos^2 \theta_L + 1)\lambda_L \delta(\omega - |\lambda_L|). \qquad (130)$$

The distance between the lines is $\lambda_L - \lambda_R = 2vt_\theta$. The relative strength of the two lines is

$$\frac{I_R(\lambda_R)}{I_L(\lambda_L)} = \sqrt{\frac{d_L^2 + h_1^2}{d_R^2 + h_1^2}}. \qquad (131)$$

The ratio is always 1 for $\beta=0$ (orthogonal orientation of DM and magnetic-field axes), see Eq. (102). Note that this is exactly the configuration in which SDW order develops, as described in detail in Sec. III B. The ordering is driven by the Cooper term, H_σ^C in Eq. (45) [equivalently, the term proportional to $[\cos(\gamma)-1]$ in (111)], which comes from the backscattering process as is made clear by the discussion in Sec. VI B. The result [Eq. (130)], obtained by neglecting backscattering, is clearly not applicable at low temperature where the SDW (Cooper) instability develops. Well below the ordering temperature the system is described by the sine-Gordon model excitations of which are massive kinks.[53] However, even above the ordering temperature in the disordered phase (which, for $\beta > 0$ extends all way down to zero temperature, see Sec. IV A) we expect the backscattering to affect the ESR signal. Whether or not it would lead to the finite linewidth of the two lines in Eq. (130) we do not understand yet and leave this interesting question (see Ref. 47 for detailed discussion of some technical subtleties) for future studies.

VIII. CONCLUSIONS

Spin-orbital interactions result in reduction of spin-rotational symmetry from $SU(2)$ to $U(1)$ in one-dimensional quantum wires and spin chains. This reduction, however, is not sufficient to change the critical (Luttinger liquid) nature of the one-dimensional interacting fermions. The situation changes dramatically once external magnetic field is applied, as we have shown in this paper. Most interesting situation occurs when the applied field is oriented along the axis orthogonal to the spin-orbital (or, Dzyaloshinskii-Moriya, in case of spin chain) axis of the wire. The resulting combination of two noncommuting perturbations, taken together with electron-electron interactions, leads to a novel spin-density-wave order in the direction of the spin-orbital axis.

The physics of this order is elegantly described in terms of spin-non-conserving (Cooper) pair-tunneling processes between Zeeman-split electron subbands. The tunneling matrix element is finite only due to the presence of the spin-orbit interaction, which allows for spin-up to spin-down (and vice versa) conversion.

The resulting SDW state affects both spin and charge properties of the wire. In particular, it suppresses effect of (weak) potential impurity, resulting in the interesting phenomena of negative magnetoresistance in one-dimensional setting, as described in Sec. V.

SDW ordering acquires true long-range nature in the case of spin chain, where charge fluctuations are absent. The staggered moment points along the DM axis and is orthogonal to the applied magnetic field.

Even when the magnetic-field and spin-orbital directions are not orthogonal, an arrangement when the critical Luttinger state survives down to the lowest temperature (see Sec. IV A), the problem remains interesting. In this geometry an ESR experiment should reveal two separate lines, which represent separate responses of right- and left-moving spin fluctuations in the system.

It is worth pointing out that unusual consequences of the interplay of spin-orbit and electron interactions are not restricted to one-dimensional systems only. We have recently shown[54] that Coulomb-coupled two-dimensional quantum dots acquire a novel van der Waals-like anisotropic interaction between spins of the localized electrons. The strength of this Ising interaction is determined by the fourth power of the Rashba coupling α_R.

We hope that our work will stimulate experimental search and studies of strongly interacting quasi-one-dimensional systems with sizable spin-orbital interaction, in particular regarding their response to the (both magnitude and direction) applied magnetic field and/or magnetization. ESR studies of spin chain materials with uniform DM interaction are very desirable as well.

ACKNOWLEDGMENTS

We would like to thank I. Affleck, L. Balents, G. Fiete, C. Kane, D. Mattis, K. Matveev, E. Mishchenko, J. Moore, L. Levitov, J. Orenstein, M. Oshikawa, M. Raikh, K. Samokhin, and Y.-S. Wu for useful discussions and suggestions at various stages of this work. O.S. thanks the Petroleum Research Fund of the American Chemical Society for the financial support of this research under the grant PRF 43219-AC10.

APPENDIX A: DERIVATION OF INTRA-SUBBAND HAMILTONIAN (29)

Noninteracting (kinetic energy) part of the νth subband ($\nu = \pm$) Hamiltonian reads

$$H^0_{\rm intra} = \frac{v_F}{2}\int dx[(\partial_x\phi_\nu)^2 + (\partial_x\theta_\nu)^2]. \tag{A1}$$

The intrasubband interaction term is given by the part of Eq. (12) which involves only densities from the νth subband

$$H'_{\rm intra} = \frac{1}{2}\int dx dx'\, U(x-x')\rho_\nu(x)\rho_\nu(x'), \tag{A2}$$

where the density in the νth subband is expressed with the help of Eq. (14) as

$$\rho_\nu = R^\dagger_\nu R_\nu + L^\dagger_\nu L_\nu + \cos[\gamma_\nu](e^{-i2k_\nu x}R^\dagger_\nu L_\nu + e^{i2k_\nu x}L^\dagger_\nu R_\nu). \tag{A3}$$

Its bosonized form follows

$$\rho_\nu = \frac{1}{\sqrt{\pi}}\partial_x\phi_\nu - \frac{\cos(\gamma_\nu)}{\pi a_0}\sin(\sqrt{4\pi}\phi_\nu + 2k_\nu x), \tag{A4}$$

where the first (second) term represents uniform ($2k_\nu$) parts of density. The interaction term (A2) then naturally splits into a sum of two contributions

$$H'_{\rm intra} = H'_0 + H'_{2k_\nu},$$

$$H'_0 = \frac{U(0)}{2\pi}\int dX(\partial_X\phi_\nu)^2, \tag{A5}$$

$$H'_{2k_\nu} = \frac{\cos^2(\gamma_\nu)}{2(\pi a_0)^2}\int dx dX U(x)\sin[\sqrt{4\pi}\phi_\nu(X+x/2) + 2k_\nu(X+x/2)]\sin[\sqrt{4\pi}\phi_\nu(X-x/2) + 2k_\nu(X-x/2)], \tag{A6}$$

where $x \to x-x'$ and $X=(x+x')/2$ are the relative and center-of-mass coordinates, respectively.

Next, following Ref. 55, we fuse the two sines in Eq. (A6) (the result is denoted as S below) using Eq. (26) and OPE identities (B1) and (B2) to obtain

$$S(x,X) = \frac{a_0^2}{4x^2}\sum_{\mu=\pm} e^{i\mu 2k_\nu x} \times\exp\{i\mu\sqrt{4\pi}[\phi(X+x/2) - \phi(X-x/2)]\}. \tag{A7}$$

Performing gradient expansion in x, neglecting boundary contribution (equivalently, using periodic boundary conditions so that $\int dX \partial_X \phi(X)=0$), and summing over $\mu=\pm 1$ leads to

$$H'_{2k_\nu} = -\frac{\cos^2(\gamma_\nu)}{2\pi}\int dx U(x)\cos(2k_\nu x) \times \int dX(\partial_X\phi_\nu)^2. \tag{A8}$$

The integral over relative distance gives backscattering component of the potential $U(2k_\nu)$. Adding two contributions we find

$$H'_{\rm intra} = \frac{U(0)-\cos^2(\gamma_\nu)U(2k_\nu)}{2\pi}\int dx(\partial_x\phi_\nu)^2. \tag{A9}$$

The sum of Eqs. (A1) and (A9) gives us the result (29).

APPENDIX B: PERTURBATIVE APPROACH TO GENERATE THE COOPER TERM

Expansion (73) relies on the following operator product expansion

$$:e^{i\alpha\phi_R(\bar z)}::e^{i\beta\phi_R(\bar z')}: = :e^{i[\alpha\phi_R(\bar z)+\beta\phi_R(\bar z')]}: \times e^{-\alpha\beta\langle\phi_R(\bar z)\phi_R(\bar z')\rangle},$$

$$:e^{i\alpha\phi_L(z)}::e^{i\beta\phi_L(z')}: = :e^{i[\alpha\phi_L(z)+\beta\phi_L(z')]}: \times e^{-\alpha\beta\langle\phi_L(z)\phi_L(z')\rangle}, \tag{B1}$$

where $z=u_\sigma\tau+ix$ and $\bar z=u_\sigma\tau-ix$ and correlation functions of chiral bosons are defined by

$$\langle\phi_L(z)\phi_L(z')\rangle = -\frac{1}{4\pi}\ln\left(\frac{z-z'}{a_0}\right),$$

$$\langle\phi_R(\bar z)\phi_R(\bar z')\rangle = -\frac{1}{4\pi}\ln\left(\frac{\bar z-\bar z'}{a_0}\right). \tag{B2}$$

Both Eqs. (B1) and (B2) follow from the harmonic S_0, see Eq. (74). We also employ Baker-Hausdorff formulae

$$e^Ae^B = e^Be^Ae^{[A,B]}, \quad e^Ae^B = e^{A+B}e^{1/2[A,B]} \tag{B3}$$

to convert expressions in terms of dual bosons $\varphi_\sigma,\theta_\sigma$ into those in terms of chiral bosons ϕ_R,ϕ_L,

$$\varphi_\sigma = \phi_L+\phi_R, \quad \theta_\sigma = \phi_L-\phi_R. \tag{B4}$$

(Note that for brevity we suppress spin index σ on the right-hand-side of the above equation.) Their commutation relations are given by Eqs. (23) and (24) with $\nu=\nu'=\sigma$.

Series [Eq. (73)] are conveniently formulated, using Eq. (B3) and $K_\sigma\to K$, in terms of

$$\begin{aligned} A_{\mu\nu}(z,\bar z) &=: e^{i\sqrt{2\pi K}\mu\varphi_\sigma}e^{i\sqrt{2\pi/K}\nu\theta_\sigma}:e^{i\mu q_0 x} \\ &= e^{i\pi/4(1/K_\sigma-K_\sigma)-i\pi/2\mu\nu} \\ &\quad\times :e^{i\sqrt{2\pi}(\mu\sqrt{K}+\nu/\sqrt{K})\phi_L(z)}e^{i\sqrt{2\pi}(\mu\sqrt{K}-\nu/\sqrt{K})\phi_R(\bar z)}:e^{i\mu q_0 x}. \end{aligned} \tag{B5}$$

Indeed,

$$\hat H_R = \frac{\tilde g_R}{4i}\int dx \sum_{\mu,\nu=\pm}\nu A_{\mu,\nu}. \tag{B6}$$

In the second-order (g_R^2) term, we need to combine $A_{\mu\nu}(z,\bar z)$ and $A_{\mu'\nu'}(z',\bar z')$. Using Eqs. (B1) and (B2), we *fuse* field $\phi_{R/L}$ at different points $z,z'=u_\sigma\tau'+ix'$ by setting $z'\to z$ everywhere where this procedure does not cause divergence. This is just an OPE-based gradient expansion. In this way we obtain

$$\begin{aligned} A_{\mu\nu}(z)A_{\mu'\nu'}(z') &= \exp\left[i\frac{\pi}{2}\left(\frac{1}{K}-K\right)-i\frac{\pi}{2}(\mu\nu+\mu'\nu')\right]\exp\left[i\frac{\pi}{2}\left(\mu\mu'K-\frac{\nu\nu'}{K}+\mu\nu'-\nu\mu'\right)\right]e^{iq_0(\mu x+\mu'x')} \\ &\quad\times :\exp\left[i\sqrt{2\pi K}(\mu+\mu')\phi_L(z)+i\sqrt{\frac{2\pi}{K}}(\nu+\nu')\phi_L(z)\right]: \times :\exp\left[i\sqrt{2\pi K}\right. \\ &\quad\left.\times(\mu+\mu')\phi_R(\bar z)-i\sqrt{\frac{2\pi}{K}}(\nu+\nu')\phi_R(\bar z)\right]: \times \exp\left[-\frac{1}{2}\ln\left(\frac{a_0}{z-z'}\right)\left(\mu\mu'K+\frac{\nu\nu'}{K}+\mu\nu'+\nu\mu'\right)\right] \\ &\quad\times\exp\left[-\frac{1}{2}\ln\left(\frac{a_0}{\bar z-\bar z'}\right)\left(\mu\mu'K+\frac{\nu\nu'}{K}-\mu\nu'-\nu\mu'\right)\right]. \end{aligned} \tag{B7}$$

In the subsequent summation over μ, μ', ν, $\nu'=\pm$ indices two combinations, with $\mu=\mu'$, $\nu=\nu'$ and $\mu=-\mu'$, $\nu=-\nu'$, produce highly irrelevant terms (of scaling dimension ≈ 4) and are disregarded easily. The choice $\mu=\mu'$ and $\nu=-\nu'$ produces *backscattering* correction

$$A_{\mu\nu}(z)A_{\mu,-\nu}(z') = \exp[i\pi(K+1/K-\mu\nu\ ] \times \left|\frac{z-z'}{a_0}\right|^{K-1/K} e^{i\mu(\sqrt{8\pi K}\varphi_\sigma+2q_0x)}. \tag{B8}$$

Finally, the choice $\mu=-\mu'$ and $\nu=\nu'$, yields the relevant *Cooper* term,

$$A_{\mu\nu}A_{-\mu\nu} = \exp[i\pi(\mu\nu - K - 1/K)] \times \left|\frac{a_0}{z-z'}\right|^{K-1/K} e^{i\mu q_0(x-x')} e^{i\nu\sqrt{8\pi/K}\theta_\sigma}. \quad \text{(B9)}$$

We then notice that for $K\equiv K_\sigma>1$ [see Eq. (41) with $\gamma_F\to 0$], the backscattering piece [Eq. (B8)] is not singular as $z'\to z$ limit is taken and simply disappears in this limit. Moreover, it is *irrelevant* (scaling dimension $2/K_\sigma>2$) and contains an oscillating phase factor $2q_0x$ which "averages" it to zero. The other, Cooper contribution [Eq. (B9)] instead *diverges* in this limit and thus has to be retained.

Collecting everything together, we find for the second-order correction $Z^{(2)}$ to the unperturbed $Z^{(0)}=\int e^{-S_0}$

$$Z^{(2)} = \frac{1}{8}\frac{\tilde{g}_R^2}{u_\sigma}f(\kappa)\int dXdT\cos[\sqrt{8\pi/K}\theta_\sigma(X,T)], \quad \text{(B10)}$$

where $(X,T)=[(x+x')/2,(\tau+\tau')/2]$ are the center-of-mass coordinates. Function $f(\kappa)$, with $\kappa\equiv(K-1/K)/2$, is given by the integral over the relative coordinates $(x,t)\to(x-x',\tau-\tau')$ [it comes from OPE result Eq. (B9)]

$$f(\kappa) = \int_{-\infty}^{\infty} dxdt\frac{\cos(q_0x)}{(x^2+t^2)^\kappa} = -2\sqrt{\pi}\cos(\pi\kappa)q_0^{2\kappa-2}\Gamma(2-2\kappa)\Gamma(\kappa-1/2)/\Gamma(\kappa),$$

$$f(\kappa\to 0) = \frac{4\pi\kappa}{q_0^2}. \quad \text{(B11)}$$

Because of its convergence, the integral can be extended to infinite limits, and this was done here. Re-exponentiating this contribution, we end up with the second-order correction $\delta S^{(2)}$ to the free action S_0

$$\delta S^{(2)} = -(\eta_\uparrow\eta_\downarrow\eta_\uparrow\eta_\downarrow)\left(\frac{\alpha_R k_F}{K_\sigma\Delta_z}\right)^2\frac{U(2k_F)}{(\pi a_0)^2K_\sigma} \times \int dXdT\cos\left(\sqrt{\frac{8\pi}{K_\sigma}}\theta_\sigma\right). \quad \text{(B12)}$$

Note that $U(2k_F)/v_F$ comes from small-κ limit of Eq. (B11) while the ratio of spin orbit to Zeeman energies appears from g_R^2/q_0^2 combination. Finally, observe that Klein factors combine to produce overall *positive* sign, as $(\eta_\uparrow\eta_\downarrow)^2=\eta_\uparrow\eta_\downarrow\eta_\uparrow\eta_\downarrow=-\eta_\uparrow^2\eta_\downarrow^2=-1$, in perfect agreement with our two-subband result in Eq. (37).

APPENDIX C: MOMENTUM SHELL RG

This appendix is intended to provide self-consistent description of the standard momentum-shell RG to the problem and to highlight few seemingly tricky technical points that arise.

Similar to the Appendix B, our starting point here are Eqs. (72)–(74). Fields $\varphi_\sigma,\theta_\sigma$ are split into *slow* (index s) and *fast* (index f) components

$$\varphi(k) = \sum_{r=s,f}\varphi_r(k), \quad \theta(k) = \sum_{r=s,f}\theta_r(k), \quad \text{(C1)}$$

where $k=(q,\omega)$ is a two momentum. Fast components have finite support only in the narrow (two-) momentum shell of "width" $d\Lambda$, and are integrated over during the RG step. Precise shape of the momentum shell to be integrated out will be discussed near the end, most of the calculation relies on the fact that the area of that shell is small, $\propto d\Lambda$. Being quadratic, the free action splits into slow $S_{0,s}$ and fast $S_{0,f}$ parts as well and, integrating out fast modes in every order of the perturbation expansion, one converts Eq. (73) into a cumulant expansion for the *effective* action

$$S_{\rm eff} = S_{0,s} - \langle S'\rangle_f - \frac{1}{2}(\langle S'^2\rangle_f - \langle S'\rangle_f^2) + \ldots, \quad \text{(C2)}$$

where the perturbation is $S'=\int d\tau\hat{H}_R$ and $\langle O\rangle_f$ stands for the expectation value of O evaluated with *fast* action $S_{0,f}$. The first-order term just produces rescaling of the coupling constant

$$\tilde{g}_R \to \tilde{g}_R' = \tilde{g}_R\times\exp\left[-\frac{1}{2}(K+1/K)\int'\frac{dk}{k}\right], \quad \text{(C3)}$$

where $\int'$ denotes integration over the fast component support area. The factor in front of the integral is just the scaling dimension of the spin-orbit operator $\hat{H}_R$, $\frac{1}{2}(K+1/K)\approx 1$. Rescaling space-time *back* produces additional factor of 2 in the exponent in Eq. (C3): $\frac{1}{2}(K+1/K)\to 2-\frac{1}{2}(K+1/K)$. This, using $\int'\frac{dk}{k}=d\Lambda/\Lambda=d\ell$, leads finally to the standard first-order RG equation for the running coupling constant, $d\tilde{g}_R/d\ell=[2-\frac{1}{2}(K+1/K)]\tilde{g}_R$. Note that Klein factors $\eta_{\uparrow,\downarrow}$ do not affect the scaling in any way. The second-order contribution contains Cooper and backscattering terms,

$$\delta S^{(2)} = \frac{(\tilde{g}_R')^2}{8}\int_1\int_2[B(r_{12})-1]\cos[\sqrt{2\pi K}(\varphi_{s,1}-\varphi_{s,2}) + q_0(x_1-x_2)]\cos\left[\sqrt{\frac{2\pi}{K}}(\theta_{s,1}+\theta_{s,2})\right] - [B^{-1}(r_{12})-1] \times\cos[\sqrt{2\pi K}(\varphi_{s,1}+\varphi_{s,2})+q_0(x_1+x_2)] \times\cos\left[\sqrt{\frac{2\pi}{K}}(\theta_{s,1}-\theta_{s,2})\right], \quad \text{(C4)}$$

where 1(20) are short-hand notations for $\vec{r}_{1,2}=(x_{1,2},\tau_{1,2})$, $\vec{r}_{12}=\vec{r}_1-\vec{r}_2$, and, for example, $\varphi_{s,1}\equiv\varphi_s(x_1,\tau_1)$. Here

$$B(r_{12}) = \exp\left[(K-1/K)\int'\frac{d^2\vec{k}}{2\pi}\frac{\cos(\vec{k}\cdot\vec{r}_{12})}{k^2}\right] \quad \text{(C5)}$$

appears from integrating out *fast* components of bosonic fields. Performing gradient expansion of Eq. (C4) we find two contributions

$$\delta S_{\rm C}^{(2)} = \frac{(\tilde{g}_R')^2}{8}B_C\int d^2\vec{R}\cos\left[\sqrt{\frac{8\pi}{K}}\theta_s(R)\right] \quad \text{(C6)}$$

$$\delta S_{\rm bs}^{(2)} = \frac{-(\tilde{g}_R')^2}{8} B_{\rm bs} \int d^2\vec{R}\, \cos[\sqrt{8\pi K}\varphi_s(R) + 2q_0 X], \tag{C7}$$

where the center-of-mass coordinates are as usual $\vec{R}=(X,T)=(\vec{r}_1+\vec{r}_2)/2$. Their amplitudes are given by

$$B_C = \int d^2\vec{r}[B(r)-1]\cos(q_0 x), \tag{C8}$$

$$B_{\rm bs} = \int d^2\vec{r}[B^{-1}(r)-1], \tag{C9}$$

and differ by the absence of *cosine* factor in the second, backscattering-related amplitude Eq. (C9). The final step is to *expand* $B(r)$ as according to its definition (C5) the expression in the exponential is proportional to small $d\Lambda$.[56] One then finds that Eq. (C8) is proportional to the product of the integral over the relative coordinate, $\int d^2\vec{r}$ *and* the integral over *fast* mode support, $\int' d^2\vec{k}$, coming from Eq. (C5). The coordinate integral is performed first to obtain

$$B_C = \pi(K-1/K)\int' d\omega dq \frac{\delta(\omega)[\delta(q-q_0)+\delta(q+q_0)]}{\omega^2+q^2}. \tag{C10}$$

We finally argue that magnetic field, which determines q_0 [Eq. (72)], breaks the symmetry between space x and time τ and this allows us to *choose* an asymmetric prescription for $\int'$. Namely, we integrate over *all* frequencies while the q integration is restricted to the $\pm|\Lambda-\Lambda'|$ interval. This gives

$$B_C = \begin{bmatrix} 2\pi(K-1/K)/q_0^2 & \text{if } q_0 \in (\Lambda-d\Lambda,\Lambda) \\ 0 & \text{otherwise.} \end{bmatrix} \tag{C11}$$

Note that the result does not contain $d\ell = d\Lambda/\Lambda$ which serves to emphasize its meaning as a fluctuation-degenerated *initial value* of the Cooper term. Let us now see what this approach predicts for the backscattering term [Eq. (C9)]. Expanding $B(r)$ as in Eq. (C10) we get

$$B_{\rm bs} = -2\pi(K-1/K)\int' d\omega dq \frac{\delta(\omega)\delta(q)}{\omega^2+q^2} + \frac{1}{2}(K-1/K)^2\int' d\omega dq \frac{1}{(\omega^2+q^2)^2} + \ldots \tag{C12}$$

While the first term is clearly zero, the second, which originates from the second-order expansion of Eq. (C5), is clearly finite for any shape of the *fast* modes support. This, *quadratic* in $U(2k_F)/v_F \ll 1$ result, is in agreement with a nonsingular structure of the similar correction found during the real-space calculation in Eq. (B8). Note finally that in the absence of magnetic field ($q_0=0$) this scheme predicts similar in structure (but different in signs) corrections to the Cooper [Eq. (C6)] and backscattering [Eq. (C7)] terms, in agreement with the result of real-space OPE calculations, see Chap. 20 in Ref. 30 and papers.[57,58]

Now we return to the original problem with finite $q_0 \neq 0$. Combining Eq. (C11) with Eq. (C6) we have

$$\delta S_C^{(2)} = -\left(\frac{\alpha_R k_F}{\Delta_z K_\sigma}\right)^2 \frac{U(2k_F)}{4(\pi a_0)^2}\int d^2\vec{R}\, \cos\left[\sqrt{\frac{8\pi}{K}}\theta_s(R)\right], \tag{C13}$$

which agrees with Eqs. (B12) and (37) in everything *but sign*! That sign comes from the Klein factors in $\tilde{g}_R$ (note that at this stage the difference between $\tilde{g}_R$ and $\tilde{g}_R'$ is of higher order in $d\ell$ and not important) and is a consequence of the identity $(\eta_\uparrow\eta_\downarrow)^2=-1$. This puzzling discrepancy between Eqs. (37) and (B12), and Eq. (C13) is worth figuring out in detail.

We note that within the functional-integral approach, which is the framework for the momentum shell RG described here, *all* information on commutation relations of dual fields φ and θ is contained in $(-i\partial_\tau\varphi\partial_x\theta)$ term in the bare action Eq. (74). This is nothing but field-theoretic version of the canonical $p\dot{x}$ term in quantum mechanics.[59] It identifies φ as a "coordinate" and $\partial_x\theta$ as a "momentum:" $[\varphi(x),\partial_{x'}\theta(x')]=i\delta(x-x')$. While fully consistent with our basic commutation relation (68), this canonical bracket *does not* contain information on the nontrivial commutation relation (23) of chiral right and left bosons. Indeed, it is a simple exercise to see that replacing Eq. (23) with *commuting* chiral bosons $[\tilde{\phi}_R,\tilde{\phi}_L]=0$ changes our (68) into $[\tilde{\varphi}(x),\tilde{\theta}(x')]=i\,\text{sign}(x'-x)/2$ which still satisfies "coordinate-momentum" identification for the pair $\tilde{\varphi}$ and $\partial_x\tilde{\theta}$. To put things differently, an analog of bosonic action [Eq. (74)] expressed in terms of "tilded" fields $\tilde{\varphi}$ and $\tilde{\theta}$ is identical to the current one in terms of our φ and θ fields satisfying Eqs. (23), (24), and (68). This simply means: bosonic action S_0, Eq. (74), does *not* enforce anticommutation relations between *right* R and *left* L moving fermions Eq. (22). This shortcoming of bosonic functional integral is well known, see for example Appendix C in Ref. 31, and several "fixes" were proposed in the literature. Our approach consists in enlarging the role of Klein factors: instead of Eq. (22) we bosonize fermions here as

$$R_s = \frac{\kappa_{Rs}}{\sqrt{2\pi a_0}}e^{i\sqrt{4\pi}\tilde{\phi}_{Rs}}, \quad L_s = \frac{\kappa_{Ls}}{\sqrt{2\pi a_0}}e^{-i\sqrt{4\pi}\tilde{\phi}_{Ls}}. \tag{C14}$$

The Klein factors $\kappa_{R/L,s}$ now carry *double* index: chirality (R or L) and spin (s). Even though the chiral bosons $\tilde{\phi}_{Rs}$ and $\tilde{\phi}_{Ls}$ now *commute*, the anticommutation of R_s and L_s is enforced by the Klein factors: $\{\kappa_{\lambda,s},\kappa_{\lambda',s'}\}=2\delta_{\lambda,\lambda'}\delta_{s,s'}$, with $\lambda=R/L$.

As a result of the proposed modification the spin-orbit term [Eq. (72)] has to be modified. It is convenient, following Ref. 60, to introduce

$$\hat{\Gamma} = \kappa_{R\uparrow}\kappa_{R\downarrow}\kappa_{L\uparrow}\kappa_{L\downarrow}. \tag{C15}$$

This product satisfies $\hat{\Gamma}^2=1$, from where it follows that its eigenvalues are $\Gamma=\pm1$. One also checks that $[\hat{\Gamma},\kappa_{R\uparrow}\kappa_{R\downarrow}]=0$ and $\kappa_{R\uparrow}\kappa_{R\downarrow}\hat{\Gamma}=-\kappa_{L\uparrow}\kappa_{L\downarrow}$. These properties allow us to represent $\hat{H}_R$ as

$$\hat{H}_R = G_R \int dx \left[\frac{(1+\hat{\Gamma})}{2} \cos(\sqrt{2\pi K}\tilde{\varphi} + q_0 x) \sin\left(\sqrt{\frac{2\pi}{K}} \tilde{\theta} \right) - \frac{(1-\hat{\Gamma})}{2} \sin(\sqrt{2\pi K}\tilde{\varphi} + q_0 x) \cos\left(\sqrt{\frac{2\pi}{K}} \tilde{\theta} \right) \right], \quad \text{(C16)}$$

$$G_R = \frac{2\alpha_R k_F}{\pi a_0} (i\kappa_{R\uparrow}\kappa_{R\downarrow}). \quad \text{(C17)}$$

Finally one observes that $\hat{\Gamma}$ *commutes* with the Hamiltonian of the problem, which implies that its eigenvalues represent integrals of motion. That is, the choice $\Gamma=+1$ or $\Gamma=-1$ is the *gauge* choice, and one can replace operator $\hat{\Gamma}$ in Eq. (C16) by its eigenvalue Γ. Comparing Eq. (C16) with our original Eq. (72) we observe that the latter corresponds to $\Gamma=+1$ gauge: with $\Gamma=+1$, Eq. (C16) transforms into Eq. (72) by replacing $(\tilde{\varphi},\tilde{\theta}) \to (\varphi_\sigma,\theta_\sigma)$. Repeating steps that led us to Eq. (C13) we arrive at

$$\delta\tilde{S}_C^{(2)} = \frac{\pi}{4}\left(\frac{G_R}{q_0}\right)^2 (K-1/K) \int d^2\vec{R} \cos\left[\sqrt{\frac{8\pi}{K}} \tilde{\theta} \right]. \quad \text{(C18)}$$

The only, but key, difference with Eq. (C13) is that now $(i\kappa_{R\uparrow}\kappa_{R\downarrow})^2 = -(\kappa_{R\uparrow}\kappa_{R\downarrow})^2 = \kappa_{R\uparrow}^2\kappa_{R\downarrow}^2 = +1$, which implies that the amplitude of the Cooper term is *positive* in Eq. (C18), in final agreement with our previous and independently derived results in Eqs. (37) and (B12). This amusing exercise illustrates the importance of Klein factors, and, more generally, of preserving correct (anti)commutation relations when implementing convenient but tricky bosonization formalism. We conclude by noting that results of the other gauge choice, $\Gamma=-1$ in Eq. (C16), while equivalent in principle to the one made above, are most conveniently understood as following from the *global* shift of bosonic fields: $\tilde{\varphi} \to \varphi + \sqrt{\pi/(8K)}$ and $\tilde{\theta} \to \theta + \sqrt{\pi K/8}$. This shift must be made in *all* bosonized expressions: it changes overall sign in Eq. (C18) but this is "compensated" by the effect of the global shift described here. As a result, the physical meaning of the Cooper instability as that of the SDW_x instability remains intact.

APPENDIX D: PERTURBATIVE APPROACH TO GENERATE THE IMPURITY TERM

The purpose of this section is to show that intersubband contribution to impurity potential, second term in Eq. (86), can be obtained as a result of interference between local impurity backscattering [first term in Eq. (86)] and bulk spin-orbit Eq. (70). Being interested in the interference between two single-particle terms, we perform the calculation directly in fermion fields and specialize to the limit $\Delta_z \gg E_{s-o}$ ($\gamma_F \to 0$) for simplicity.

The lowest order in the perturbative expansion involving these two terms is obtained by the following correction to partition function [compare with Eq. (73)]

$$\delta Z_{\text{imp}} = \frac{1}{2}\int e^{-S_0} \int d\tau d\tau' \hat{H}_R(\tau) V^B(\tau') \to \int e^{-S_0} \int d\tau \delta V^B, \quad \text{(D1)}$$

where the second line identifies correction to the impurity backscattering term we are after. Here S_0 is the action of two Zeeman-split $\{\uparrow,\downarrow\}$ subbands, and V^B is the intrasubband term due to the impurity, located at $x=0$, given by

$$V^B = V_0 \sum_{s=\uparrow,\downarrow} (R_s^\dagger L_s + L_s^\dagger R_s)_{x=0}. \quad \text{(D2)}$$

The spin-orbit term $\hat{H}_R$ in the presence of magnetic field (so that $k_\uparrow - k_\downarrow = \delta k_F$) reads

$$\hat{H}_R = \alpha_R k_F \int dx \sum_{s=\uparrow,\downarrow} (R_s^\dagger R_{-s} e^{-is\delta k_F x} - L_s^\dagger L_{-s} e^{is\delta k_F x}), \quad \text{(D3)}$$

where we neglected terms $\propto \delta k_F/k_F = \Delta_z/E_F \ll 1$.

We calculate Eq. (D1) by fusing fermions with such as spin index, that is by making the replacement $\Psi_{R/Ls}(x,\tau)\Psi^\dagger(x',\tau') \to G_{R/L}(x-x',\tau-\tau')$ where possible. Here $G_{R/L}$ stands for fermions Green's functions. To lowest (zeros) order in the interaction these are given by the *free* fermion Green's functions[41]

$$G_R(x,\tau) = \langle \Psi_{Rs}(x,\tau)\Psi_{Rs'}^\dagger(0)\rangle = \frac{\delta_{s,s'}}{2\pi(v\tau - ix)}$$

$$G_L(x,\tau) = \langle \Psi_{Ls}(x,\tau)\Psi_{Ls'}^\dagger(0)\rangle = \frac{\delta_{s,s'}}{2\pi(v\tau + ix)}. \quad \text{(D4)}$$

In this way we find

$$\int d\tau \delta V^B = \frac{\alpha_R k_F V_0}{4\pi} \int d\tau [(R_\uparrow^\dagger L_\downarrow - L_\uparrow^\dagger R_\downarrow) P_1 + (R_\downarrow^\dagger L_\uparrow - L_\downarrow^\dagger R_\uparrow) P_2], \quad \text{(D5)}$$

where

$$P_1 = \int dx dt \left(\frac{e^{-i\delta k_F x}}{\bar{z}} + \frac{e^{i\delta k_F x}}{z} \right) \quad \text{(D6)}$$

$$P_2 = \int dx dt \left(\frac{e^{i\delta k_F x}}{\bar{z}} + \frac{e^{-i\delta k_F x}}{z} \right), \quad \text{(D7)}$$

and $z = v_F t + ix$, $\bar{z} = v_F t - ix$ as usual. These integrals are easily calculated

$$P_1 = -P_2 = \frac{4\pi}{\delta k_F v_F^2}. \quad \text{(D8)}$$

Expressing $\delta k_F = \Delta_z / v_F$ we obtain the correction

$$\delta V^B = \frac{\alpha_R k_F V_0}{\Delta_z}(R_\uparrow^\dagger L_\downarrow - R_\downarrow^\dagger L_\uparrow + L_\downarrow^\dagger R_\uparrow - L_\uparrow^\dagger R_\downarrow). \quad \text{(D9)}$$

We observe that δV^B is *odd* under spatial inversion $\mathcal{P}$ (with respect to impurity location) when $x \to -x$ and right and left movers get interchanged, $R_s \leftrightarrow L_s$. Of course, it *must* be odd under $\mathcal{P}$ as it is obtained from fusing *even* Eq. (D2) and *odd* Eq. (D3)in $\mathcal{P}$ terms. The oddness of Eq. (D9) is the reason for the relative minus signs in this equation.

Bosonization of Eq. (D9), following Sec. III A, results in

$$\delta V^B = \frac{2\alpha_R k_F V_0}{\Delta_z}\frac{i\eta_\uparrow\eta_\downarrow}{\pi a_0}\sin(\sqrt{2\pi}\varphi_\rho)\cos(\sqrt{2\pi}\theta_\sigma), \quad \text{(D10)}$$

which confirms our previous result [Eqs. (85) and (86)]. The generated impurity potential describes *intra*subband impurity backscattering.

The absence of the potentially dangerous term with $\sin(\sqrt{2\pi}\theta_\sigma)$ in place of $\cos(\sqrt{2\pi}\theta_\sigma)$ in Eq. (D10) is now clear: such a term would require no minus signs in Eq. (D9) which is forbidden by the oddness of δV^B under inversion $\mathcal{P}$.

[1] S. Datta and B. Das, Appl. Phys. Lett. **56**, 665 (1990).
[2] Yu. A. Bychkov and E. I. Rashba, J. Phys. C **17**, 6039 (1984).
[3] A. V. Moroz and C. H. W. Barnes, Phys. Rev. B **60**, 14272 (1999); **61**, R2464 (2000).
[4] P. Streda and P. Seba, Phys. Rev. Lett. **90**, 256601 (2003).
[5] L. S. Levitov and E. I. Rashba, Phys. Rev. B **67**, 115324 (2003).
[6] R. G. Pereira and E. Miranda, Phys. Rev. B **71**, 085318 (2005).
[7] H. A. Engel, E. I. Rashba, and B. I. Halperin, *Handbook of Magnetism and Advanced Magnetic Materials* (Wiley, New York, 2007), Vol. 5; *Handbook of Magnetism and Advanced Magnetic Materials* (Wiley, New York, in press), Vol. 5.
[8] T. Kimura, K. Kuroki, and H. Aoki, Phys. Rev. B **53**, 9572 (1996).
[9] W. Häusler, Phys. Rev. B **63**, 121310(R) (2001).
[10] A. V. Moroz, K. V. Samokhin, and C. H. W. Barnes, Phys. Rev. B **62**, 16900 (2000).
[11] A. Iucci, Phys. Rev. B **68**, 075107 (2003).
[12] M. Governale and U. Zülicke, Solid State Commun. **131**, 581 (2004).
[13] Y. Yu, Y. Wen, J. Li, Z. Su, and S. T. Chui, Phys. Rev. B **69**, 153307 (2004).
[14] V. Gritsev, G. I. Japaridze, M. Pletyukhov, and D. Baeriswyl, Phys. Rev. Lett. **94**, 137207 (2005).
[15] H. C. Lee and S. R. Eric Yang, Phys. Rev. B **72**, 245338 (2005).
[16] T. Hikihara, A. Furusaki, and K. A. Matveev, Phys. Rev. B **72**, 035301 (2005).
[17] G. Dresselhaus, Phys. Rev. **100**, 580 (1955).
[18] A. Mugarza, A. Mascaraque, V. Repain, S. Rousset, K. N. Altmann, F. J. Himpsel, Y. M. Koroteev, E. V. Chulkov, F. J. Garcia de Abajo, and J. E. Ortega, Phys. Rev. B **66**, 245419 (2002).
[19] J. N. Crain and F. J. Himpsel, Appl. Phys. A: Mater. Sci. Process. **82**, 431 (2006).
[20] A. Mugarza and J. E. Ortega, J. Phys.: Condens. Matter **15**, S3281 (2003).
[21] J. E. Ortega, M. Ruiz-Osés, J. Gordón, A. Mugarza, J. Kuntze, and F. Schiller, New J. Phys. **7**, 101 (2005).
[22] S. LaShell, B. A. McDougall, and E. Jensen, Phys. Rev. Lett. **77**, 3419 (1996).
[23] O. Krupin, G. Bihlmayer, K. Starke, S. Gorovikov, J. E. Prieto, K. Döbrich, S. Blügel, and G. Kaindl, Phys. Rev. B **71**, 201403(R) (2005).
[24] Yu. S. Dedkov, M. Fonin, U. Rüdiger, and C. Laubschat, Phys. Rev. Lett. **100**, 107602 (2008).
[25] J. Sun, S. Gangadharaiah, and O. A. Starykh, Phys. Rev. Lett. **98**, 126408 (2007).
[26] D. C. Dender, P. R. Hammar, D. H. Reich, C. Broholm, and G. Aeppli, Phys. Rev. Lett. **79**, 1750 (1997).
[27] I. Affleck and M. Oshikawa, Phys. Rev. B **60**, 1038 (1999).
[28] D. V. Zakharov, J. Deisenhofer, H.-A. Krug von Nidda, P. Lunkenheimer, J. Hemberger, M. Hoinkis, M. Klemm, M. Sing, R. Claessen, M. V. Eremin, S. Horn, and A. Loidl, Phys. Rev. B **73**, 094452 (2006).
[29] O. A. Starykh, D. L. Maslov, W. Häusler, and L. I. Glazman, Lect. Notes Phys. **544**, 37 (2000).
[30] A. O. Gogolin, A. A. Nersesyan, and A. M. Tsvelik, *Bosonization and Strongly Correlated Systems* (Cambridge University Press, Cambridge, England, 1999).
[31] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, New York, 2004).
[32] H.-H. Lin, L. Balents, and M. P. A. Fisher, Phys. Rev. B **56**, 6569 (1997).
[33] T. Giamarchi and H. J. Schulz, J. Phys. (Paris) **49**, 819 (1988).
[34] C. L. Kane and M. P. A. Fisher, Phys. Rev. B **46**, 15233 (1992).
[35] E. Orignac and T. Giamarchi, Phys. Rev. B **56**, 7167 (1997).
[36] R. Egger and A. O. Gogolin, Eur. Phys. J. B **3**, 281 (1998).
[37] C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 226801 (2005).
[38] C. Xu and J. E. Moore, Phys. Rev. B **73**, 045322 (2006).
[39] C. Wu, B. A. Bernevig, and S.-C. Zhang, Phys. Rev. Lett. **96**, 106401 (2006).
[40] S. Eggert, Phys. Rev. B **54**, R9612 (1996).
[41] O. A. Starykh, A. Furusaki, and L. Balents, Phys. Rev. B **72**, 094416 (2005).
[42] M. Bocquet, F. H. L. Essler, A. M. Tsvelik, and A. O. Gogolin, Phys. Rev. B **64**, 094425 (2001).
[43] J.-S. Caux, F. H. L. Essler, and U. Löw, Phys. Rev. B **68**, 134431 (2003).
[44] A. P. Schnyder, O. A. Starykh, and L. Balents, arXiv:0807.0285 (unpublished).
[45] M. Oshikawa, K. Ueda, H. Aoki, A. Ochiai, and M. Kohgi, J. Phys. Soc. Jpn. **68**, 3181 (1999).
[46] M. Oshikawa and I. Affleck, Phys. Rev. Lett. **82**, 5136 (1999).
[47] M. Oshikawa and I. Affleck, Phys. Rev. B **65**, 134410 (2002).
[48] R. Kubo and K. Tomita, J. Phys. Soc. Jpn. **9**, 888 (1954).
[49] H. Mori and K. Kawasaki, Prog. Theor. Phys. **27**, 529 (1962).
[50] H. Mori and K. Kawasaki, Prog. Theor. Phys. **28**, 971 (1962).
[51] A. De Martino, R. Egger, K. Hallberg, and C. A. Balseiro, Phys. Rev. Lett. **88**, 206402 (2002).
[52] A. De Martino, R. Egger, F. Murphy-Armando, and K. Hallberg, J. Phys.: Condens. Matter **16**, S1437 (2004).
[53] S. A. Zvyagin, A. K. Kolezhuk, J. Krzystek, and R. Feyerherm,

Phys. Rev. Lett. **95**, 017207 (2005).
[54] S. Gangadharaiah, J. Sun, and O. A. Starykh, Phys. Rev. Lett. **100**, 156402 (2008).
[55] S. Capponi, D. Poilblanc, and T. Giamarchi, Phys. Rev. B **61**, 13410 (2000).
[56] We thank L. Balents for sharing this trick.
[57] A. A. Nersesyan, A. Luther, and F. V. Kusmartsev, Phys. Lett. A **176**, 363 (1993).
[58] V. M. Yakovenko, JETP Lett. **56**, 5101 (1992).
[59] X. G. Wen, *Quantum Field Theory of Many-Body Systems* (Oxford University Press, New York, 2004).
[60] J. O. Fjærestad and J. B. Marston, Phys. Rev. B **65**, 125106 (2002).